1D Cubic NLS (CNLS) and Cubic-Quintic NLS (CQNLS)

Marcello Sani

December 5, 2006

Contents

Introduction

The nonlinear cubic-quintic Schrödinger equation (CQNLS) is the following differential equation:

$$\imath u_t + u_{xx} + a_1 u \left|u\right|^2 - a_2 u \left|u\right|^4 = 0,$$

where a_1 and a_2 are real constants.

It takes its name from the fact that the small amplitude approximation is the equation that Schrödinger proposed in the year 1926 for the propagation of a quantum wave packet in free space.

Physical applications of the CNLS

The CNLS is a generic equation, arising whenever one studies unidirectional propagation of wave packets in a dispersive energy conserving medium at the lowest order of nonlinearity.

Applications of the CNLS are

- Description of non-linear pulses on an optical fiber
- Two-dimensional self-focusing of a plane wave
- One-dimensional self-modulation of a monochromatic wave
- Propagation of a heat pulse in a solid
- Langmuir waves in plasmas.

From the Sine-Gordon to the 1D CNLS

Start from the Sine-Gordon equation

$$u_{tt} - c_0^2 u_{xx} + \omega_0^2 \sin u = 0.$$

We consider only the first two terms of the Taylor-development of the sinus function

$$\sin u = u - \frac{u^3}{6} + \dots$$

A first idea could be to look for a solution of the form of a plane wave with a small perturbating term

$$u(x,t) = \varepsilon A e^{\imath(qx - \omega t)} + \varepsilon^2 B(x,t),$$

but in this case we would obtain $B(x,t) \sim \varepsilon^2 t$, i.e. $B(x,t)$ diverges.

Introduce multiple scales expansion

$$T_i = \varepsilon^i t, \qquad X_i = \varepsilon^i x,$$

so the solution will be of the form

$$u(x,t) = \varepsilon \sum_{i=0}^{\infty} \varepsilon^i \phi_i \left(X_0, X_1, X_2, ..., T_0, T_1, T_2, ... \right).$$

We use the notation

$$D_i = \frac{\partial}{\partial T_i}, \qquad D_{X_i} = \frac{\partial}{\partial X_i}.$$

At the order ε we need to solve

$$\left(D_0^2 - c_0^2 D_{X_0}^2 + \omega_0^2\right)\phi_0 = 0.$$

The solution is a plane wave:

$$\phi_0 = A(X_1, T_1, X_2, T_2, ...)e^{\imath(qX_0 - \omega T_0)} + c.c.,$$

where c.c. is the complex conjugate.
It holds the dispersion relation $\omega^2 = \omega_0^2 + c_0^2 q^2$.

At the order ε^2 we find

$$D_0^2\phi_1 + 2D_0D_1\phi_0 - c_0^2 D_{X_0}^2\phi_1 - 2c_0^2 D_{X_0}D_{X_1}\phi_0 + \omega_0^2\phi_1 = 0.$$

We need the following condition to eliminate the secular terms

$$\frac{\partial A}{\partial T_1} + \frac{qc_0^2}{\omega}\frac{\partial A}{\partial X_1} = 0,$$

where $v_g := \frac{qc_0^2}{\omega}$ is the group velocity.
The solution is therefore

$$\phi_1 = 0.$$

Order ε^3:

$$-D_1^2\phi_0 - 2D_0D_2\phi_0 + c_0^2 D_{X_1}^2\phi_0 + 2c_0^2 D_{X_0}D_{X_2}\phi_0 + \frac{\omega_0^2}{6}\phi_0^3$$

$$-2D_0D_1\phi_1 + 2c_0^2 D_{X_0}D_{X_1}\phi_1 = 0.$$

As above we need the condition

$$-\frac{\partial^2 A}{\partial T_1^2} + 2\imath\omega\frac{\partial A}{\partial T_2} + c_0^2\frac{\partial^2 A}{\partial X_1^2} + 2\imath q c_0^2\frac{\partial A}{\partial X_2} + \frac{3}{6}\omega_0^2 \left|A\right|^2 A = 0.$$

Introducing the new variables

$$\xi_i = X_i - v_g T_i, \qquad \tau_i = T_i$$

and using the condition of the order ε^2, we have

$$\left(c_0^2 - v_g^2\right)\frac{\partial^2 A}{\partial \xi_1^2} + 2\imath\omega\left(\frac{\partial A}{\partial \tau_2} - v_g\frac{\partial A}{\partial \xi_2}\right) + 2\imath q c_0^2\frac{\partial A}{\partial \xi_2} + \frac{1}{2}\omega_0^2\left|A\right|^2 A = 0.$$

Using the expression for the group velocity

$$v_g = qc_0^2/\omega,$$

we obtain

$$\imath\frac{\partial A}{\partial \tau_2} + \frac{\left(c_0^2 - v_g^2\right)}{2\omega}\frac{\partial^2 A}{\partial \xi_1^2} + \frac{\omega_0^2}{4\omega}\left|A\right|^2 A = 0,$$

which is just the CNLS with

$$a_1 = \frac{\omega_0^2}{4\omega}\frac{2\omega}{\left(c_0^2 - v_g^2\right)} = \frac{\omega_0^2}{2\left(c_0^2 - v_g^2\right)}.$$

Single soliton solution for 1D CNLS

This is found by looking for solutions of the CNLS, with $a_1 = \nu$, depending on a moving coordinate $X = x - Ut$:

$$u = e^{\imath rx - \imath st} v(X), \qquad X = x - Ut,$$

where r and s are constants.
On substitution, the ordinary differential equation for v is

$$v'' + \imath(2r - U)v' + (s - r^2)v + \nu |v|^2 v = 0.$$

We now choose

$$r = \frac{U}{2} \text{ and } s = \frac{U^2}{4} - \alpha,$$

the first being the important one to eliminate the term in v'.

Then v may be taken to be real and

$$v'' - \alpha v + \nu v^3 = 0.$$

This gives rise to a cnoidal wave equation for v. It may be integrated once to

$$v'^2 = A + \alpha v^2 - \frac{\nu}{2} v^4,$$

which can be solved in elliptic functions.
The limiting case of the solitary wave is possible when $\nu > 0$; we take $A = 0$, $\alpha > 0$, and the solution is

$$v = \left(\frac{2\alpha}{\nu}\right)^{1/2} \operatorname{sech}\left(\alpha^{1/2}(x - Ut)\right).$$

Analytical solutions of the CQNLS

As seen it is possible to find analytic solutions of the CQNLS. One way of doing this is the following: we consider the differential equation in the form

$$\imath u_t + u_{xx} = a_1 u \left|u\right|^2 + a_2 u \left|u\right|^4$$

and we use the "Ansatz"

$$u(x,t) = f(x) e^{-\imath a t},$$

where a is a real constant and f a complex function. The CQNLS reduces to

$$f_{xx} + af = a_1 f \left|f\right|^2 + a_2 f \left|f\right|^4 .$$

We now set

$$f(x) = M(x)e^{\imath N(x)},$$

where M and N are real functions. Separating the real and the imaginary part of $f(x)$, we get

$$M_{xx} - M\left(N_x\right)^2 + aM = a_1 M^3 + a_2 M^5$$

and

$$2M_x N_x + M N_{xx} = 0.$$

Multiplying the second equation by M

$$2MM_xN_x + M^2N_{xx} = 0,$$

i.e.

$$\left(M^2\right)_x N_x + M^2N_{xx} = 0,$$

and integrating twice we obtain N in terms of M

$$N = S\int M^{-2}dx + N_0,$$

where S and N_0 are real integration constants. N_0 represents a constant change of phase.

Substituting this last equation into the first, leads to

$$M_{xx} - S^2M^{-3} + aM = a_1M^3 + a_2M^5.$$

We have now to multiply by M_x and integrate to get

$$(M_x)^2 + S^2 M^{-2} + aM^2 = a_1 \frac{M^4}{2} + a_2 \frac{M^6}{3} + K_0,$$

where K_0 is an integration constant. This equation leads to a standard elliptic integral by the substitution

$$M(x) = [pW(y)]^{1/2}, \quad pW > 0,$$

where p is a nonvanishing constant and

$$y = \left(\frac{pa_1}{2}\right)^{1/2} x, \quad \text{for } a_2 = 0,$$

$$y = \left(\frac{4a_2}{3}\right)^{1/2} px, \quad \text{for } a_2 \neq 0.$$

Then we have for the cubic case

$$(W_y)^2 = 4W^3 - \frac{8a}{a_1 p} W^2 + 4KW - \frac{8S^2}{a_1 p^3}$$
$$\equiv 4(W - W_1)(W - W_2)(W - W_3)$$

and for the quintic case

$$(W_y)^2 = W^4 + \frac{3a_1}{2a_2 p} W^3 - \frac{3a}{a_2 p^2} W^2 + 4KW - \frac{3S^2}{a_2 p^4}$$
$$\equiv (W - W_1)(W - W_2)(W - W_3)(W - W_4),$$

where $K := pK_0 y^2/x^2$ is a constant and the W_i's are the roots of the right-hand sides.

An example of analytical solution of the CQNLS

The equation

$$\imath u_t + u_{xx} + u\left|u\right|^2 - \delta u\left|u\right|^4 = 0$$

has the solution

$$u(x,t) = \sqrt{2}\frac{\operatorname{sech} x}{\left[C + 1 - \frac{C}{2}\operatorname{sech}^2 x\right]^{1/2}} e^{\imath t},$$

for

$$C = \frac{4\delta W_3}{3} = -\frac{4\delta}{3}\left(\frac{3}{4\delta} - \frac{1}{2}\sqrt{\frac{3}{4\delta^2(3-16\delta)}}\right)$$

$$= -1 + \frac{2\delta}{3}\sqrt{\frac{3}{4\delta^2(3-16\delta)}}.$$

Where W_3 is one of the roots of the polynomial

$$W^4 + \frac{3}{2\delta}W^3 + \frac{3}{\delta}W^2,$$

i.e.

$$W_1 = 0, \qquad W_2 = 0$$

$$W_3 = -\frac{3}{4\delta} + \frac{1}{2}\sqrt{D}, \qquad W_4 = -\frac{3}{4\delta} - \frac{1}{2}\sqrt{D},$$

with

$$D = \frac{3}{4}\frac{3 - 16\delta}{\delta^2}.$$

Note that the constant C is complex for $\delta > 3/16$.
In fact $D < 0$ for $\delta > \frac{3}{16} = 0.1875$, because

$$D = \frac{3}{4}\frac{3-16\delta}{\delta^2} < 0$$

$$\Rightarrow \frac{3-16\delta}{\delta^2} < 0$$

$$\Rightarrow 3-16\delta < 0$$

$$\Rightarrow \delta > \frac{3}{16}.$$

Numerical computation of a solution of the CQNLS

We look for a solution of the CQNLS

$$\imath u_t + u_{xx} + u\left|u\right|^2 - \delta u\left|u\right|^4 = 0,$$

for $\delta = 1$, using a known solution,

$$z(x,t) = \sqrt{2}\,\mathrm{sech}(x)e^{\imath t},$$

of the CNLS ($\delta = 0$).
We perform numerical homotopy continuation of $z(x,t)$ by slowly increasing the value of δ up to 1.
Suppose at first the wanted solution has the form

$$u(x,t) = v(x)e^{\imath t},$$

for a real function $v(x)$ that we need to determine numerically.

This leads to the following differential equation for v

$$-v + v'' + v^3 - \delta v^5 = 0,$$

that we discretize with the finite difference method and solve, at each value of δ, using the Newton iteration.
We are starting from the function

$$v_0(x) := z(x, 0) = \sqrt{2}\,\mathrm{sech}(x)$$

that sastisfies the differential equation

$$-v_0 + v_0'' + v_0^3 - \delta v_0^5 = 0.$$

As yet seen, analitically the obtained solution would be

$$u(x,t) = \sqrt{2}\frac{\mathrm{sech}(x)}{\left[C + 1 - C/2\,\mathrm{sech}^2(x)\right]^{1/2}} e^{\imath t},$$

for a constant C depeding on δ.

Moving waves

The above numerically calculated wave is a stationary one. We are now going to see how it is possible to derive a moving one. We assume

$$u(x,t) = Ae^{\imath\omega t}v(x)$$

solves the CQNLS

$$\imath u_t + u_{xx} + u\left|u\right|^2 - \delta u\left|u\right|^4 = 0.$$

It follows that v satisfies

$$-A\omega v + Av'' + A^3v^3 - \delta A^5 v^5 = 0.$$

Use Galilean boost to obtain a moving wave, with speed w, starting from a stationary one. Let

$$\tilde{x} := x - wt$$

$$\tilde{t} := t.$$

We would like to know how frequency and amplitude get transformed

$$\tilde{\omega} = ?$$

$$\tilde{A} = ?$$

We define

$$\tilde{u}(x,t) := \tilde{A}(w)e^{\imath\left(\tilde{\omega}(w)\tilde{t}+\mu x\right)}v(\tilde{x}) = \tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v(x-wt),$$

for a constant μ.
Inserting, we get a new differential equation for v

$$\imath\left(\imath\tilde{\omega}(w)\tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v - w\tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v'\right)+\tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v''$$

$$+2\imath\mu\tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v' - \mu^2\tilde{A}(w)e^{\imath(\tilde{\omega}(w)t+\mu x)}v + \tilde{A}^3(w)v^3e^{\imath(\tilde{\omega}(w)t+\mu x)}$$
$$-\delta\tilde{A}^5(w)v^5e^{\imath(\tilde{\omega}(w)t+\mu x)} = 0.$$

It follows

$$\tilde{A}(w)v'' - \tilde{A}(w)\tilde{\omega}(w)v - \imath\tilde{A}(w)wv' + 2\imath\tilde{A}(w)\mu v' - \tilde{A}(w)\mu^2 v$$

$$+\tilde{A}^3(w)v^3 - \delta\tilde{A}^5(w)v^5 = 0.$$

Taking now $w = 2\mu$, we get

$$v'' - \tilde{\omega}(w)v - \mu^2 v + \tilde{A}^2(w)v^3 - \delta\tilde{A}^4(w)v^5 = 0.$$

So $\tilde{u}$ is still a solution of the CQNLS if

$$\tilde{\omega} = \omega - \mu^2 = \omega - \frac{w^2}{4}$$

$$\tilde{A} = A.$$

Then

$$\tilde{u}\,(x,t) = Ae^{\imath\ \ (\omega - \frac{w^2}{4}) + \frac{w}{2}x}\ \ v(x - wt).$$

also solves the CQNLS.

References

- Cagnon, L.: *Exact travelling-wave solutions for optical models based on the nonlinear cubic-quintic Schrödinger equation*. JOSA A. **6**, 1477-1483 (1989)
- Peyard, M., Dauxois, T.: *Physique des solitons*. Les Ullis: EDP Eciences, 72-76 (2004)
- Scott, A.: *Nonlinear science*. Oxford Univ. Press, Oxford, 56 (2003)
- Whitham G. B.: *Linear and Nonlinear waves*. Wiley-Interscience, Sec. 17.8, (1974)