Convergence of Petviashvili's Iteration Method

Peter Kauf

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Numbering consistent with [PS] !

Outline

Scalar, 1-D Wave Equation with Power Nonlinearity

$$u_t - (\mathcal{L}\, u)_x + p u^{p-1} u_x = 0\ , \tag{1.1}$$

- $u : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, $p > 1$
- $\mathcal{L}$: linear, self-adjoint ($\langle u, \mathcal{L}\, v\rangle = \langle \mathcal{L}\, u, v\rangle$), positive ($\langle u, \mathcal{L}\, u\rangle \geq 0$) pseudodifferential operator in x of order m.
- $\langle f, g\rangle = \int_{-\infty}^{\infty} \bar{f}(x) g(x) dx$
- Fourier: $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k) e^{ikx} dk$, $\hat{u}(k) = \int_{-\infty}^{\infty} u(x) e^{-ikx} dx$

Stationary bound state solution $u(x,t) = \Phi(x - ct)$ leads to boundary value problem ($\int \left[-c\Phi_x - (\mathcal{L}\, \Phi)_x + p\Phi^{p-1}\Phi_x\right] dx$)

$$(1.3) \begin{cases} c\Phi + \mathcal{L}\, \Phi = \Phi^p \\ \lim\limits_{|x|\to\infty} \Phi(x) = 0 \end{cases} \quad \text{or } (1.5)\ \ [c + v(k)]\, \widehat{\Phi}(k) = \widehat{\Phi^p}(k)\ ,$$

$v(k) \geq 0$ an m^{th} order polynomial in $|k|$

Assumption, Solution Space, Iteration

Assumption 1.1

$p > 1$, $v(k) \geq 0$, $c > 0$. $\exists$ real analytical solution to 1 in

$$X = L^2(\mathbb{R}) \cap L^{p+1}(\mathbb{R}) \cap H^{m/2}(\mathbb{R})$$

Approximate $\widehat{\Phi}$ through $\widehat{u}_{n+1}(k) = \frac{\widehat{u_n^p}(k)}{c+v(k)} \longrightarrow$ usually divergent !

Petviashvili Iteration

Well defined with assumption 1.1 !

$$\widehat{u}_{n+1}(k) = M_n^\gamma \frac{\widehat{u_n^p}(k)}{c + v(k)} \qquad (1.8)$$

$$M_n = \frac{\int_{-\infty}^{\infty} [c + v(k)][\widehat{u}_n(k)]^2 dk}{\int_{-\infty}^{\infty} \widehat{u}_n(k)\widehat{u_n^p}(k) dk} \qquad (1.9)$$

Lemma 1.2: Fix points for (1.8), (1.9) correspond to bound states $\widehat{\Phi}(k)$ of (1.5) for $\gamma \neq 1 + 2n$, $n \in \mathbb{Z}$.

Spectrum, Assumption 2.1

Define Operator to (1.1): $\mathcal{H} = c + \mathcal{L} - p\Phi^{p-1}(x)$ (1.10)

- selfadj. in $L^2(\mathbb{R}) \longrightarrow$ real eigenval., orth. spectr. decomp.
- Null space contains at least $\Phi'(x)$.
- cont. spectrum positive, bounded away from zero (ass. 1.1)
- negative spectrum not empty

$$\begin{aligned}
\mathcal{H}\,\Phi &= (1-p)\Phi^p \\
\langle \mathcal{H}\,\Phi, \Phi \rangle &= -(p-1)\langle \Phi^p, \Phi \rangle = -\frac{p-1}{2\pi}\langle \widehat{\Phi}, \widehat{\Phi^p} \rangle \\
&= -\frac{p-1}{2\pi}\langle [c+v(.)]\widehat{\Phi}, \widehat{\Phi} \rangle < 0
\end{aligned}$$

Spectrum, Assumption 2.1

Define Operator to (1.1): $\mathcal{H} = c + \mathcal{L} - p\Phi^{p-1}(x)$ (1.11)

- selfadj. in $L^2(\mathbb{R})$ $\longrightarrow$ real eigenval., orth. spectr. decomp.
- Null space contains at least $\Phi'(x)$.
- cont. spectrum positive, bounded away from zero (ass. 1.1)
- negative spectrum not empty

Assumption 2.1 on Spectrum of $\mathcal{H}$:

- $\sigma_{L^2}^{\text{discr}}(\mathcal{H})$ for eigenvalues $< c$
- $\sigma_{L^2}^{\text{cont}}(\mathcal{H})$ for eigenvalues $\geq c$
- Nullspace is one-dimensional
- dim. neg. space $n(\mathcal{H}) \geq 1$

Outline

Convergence Theorem

Theorem 2.8

Let $\widehat{\Phi}(k)$ solution to (1.5), assumptions 1.1 and 2.1. Petviashvili Iteration (1.8), (1.9) converges to $\widehat{\Phi}(k)$ in (small) neighbourhood of $\widehat{\Phi}(k)$ if:

1. $1 < \gamma < \frac{p+1}{p-1}$
2. $n(\mathcal{H}) = 1$
3. Either $\Phi^{p-1}(x) \geq 0$ or $\lambda_{\max}((c+\mathcal{L})^{-1}\mathcal{H}) < 2$ (ass. 2.7)

"If any of the conditions are not met, the Petviashvili iteration diverges from $\widehat{\Phi}(k)$".

Fréchet Derivative, Contraction Principle

Fréchet Derivative

$\mathcal{B}$, $\mathcal{C}$ Banach spaces, $D \subset \mathcal{B}$ open, mapping $A : \mathcal{B} \longrightarrow \mathcal{C}$. A is Fréchet differentiable in $g \in D$ if $\exists$ linear operator $L : \mathcal{B} \longrightarrow \mathcal{C}$, such that

$$\lim_{\|h\| \to 0} \frac{\|A(g+h) - Ag - Lh\|}{\|h\|} = 0$$

Fixed Point Theorem ([HP], Lemma 4.4.8)

Let $\mathcal{B}$ a Banach space, $D \subset \mathcal{B}$ open, assume that $A : D \longrightarrow \mathcal{B}$ has fixed point $\bar{f} \in D$, and let A Fréchet diff. in $\bar{f}$ $(A'(\bar{f}))$.

$\forall \quad 0 < \varepsilon < 1 - \|A'(\bar{f})\| \quad \exists S(\bar{f}, \delta)$ open such that if $f_0 \in S(\bar{f}, \delta)$:

- The iterates $f_n := Af_{n-1} \in S(\bar{f}, \delta)$
- $\lim f_n = f$
- $\|f_n - \bar{f}\| \leq (\|A'(f)\| + \varepsilon)^n \, \|f_0 - f\|$

Proof of Convergence

Let A the iteration operator (1.8), (1.9): $\hat{u}_{n+1} = A(\hat{u}_n)$ in $X(\mathbb{R})$.

1. $A'(\hat{u}_n)$ continuous in $S(\widehat{\Phi}, \delta_c)$ (proof: [PS], Proposition 3.4 and additional calculation)
2. $\|A'(\widehat{\Phi})\| < 1$, i.e. spectral radius of $A'(\widehat{\Phi})$ is < 1.

By Continuity of $A'(\widehat{\Phi})$, we have $\forall \quad 0 < \varepsilon < 1 - \|A'(\widehat{\Phi})\|$ $\exists S(\widehat{\Phi}, \delta(\varepsilon)) \subset X(\mathbb{R})$ such that $q = \sup_{\hat{u}_n \in S} \|A'(\hat{u}_n)\| < 1$.

By [HP], Lemma 4.4.7: $\forall \quad \hat{f}, \hat{g} \in S$: $\|A(\hat{f}) - A(\hat{g})\| \leq q\|\hat{f} - \hat{g}\|$. The contraction mapping theorem ([HP] theorem 4.3.4) assures that $A(\hat{u}_n)$ has **unique**, asymptotically stable fixed point in $S(\widehat{\Phi}, \delta)$. By the fixed Point theorem we get that

$$\|\hat{u}_n - \widehat{\Phi}\| \leq \left(\|A'(\widehat{\Phi})\| + \varepsilon\right)^n \|\hat{u}_0 - \widehat{\Phi}\|.$$

q.e.d. theorem 2.8

Outline

Proposition 3.1

Proposition 3.1 $A'(\widehat{\Phi})$ (i.e. Operator (1.8), (1.9) linearized at $\widehat{\Phi}(k)$) has spectral radius smaller than one ($\|A'(\widehat{\Phi})\| < 1$), if

- $1 < \gamma < \frac{p+1}{p-1}$
- $n(\mathcal{H}) = 1$
- assumptions 2.1 and 2.7 are met.

Proof: Define $\hat{u}_0(k) := \widehat{\Phi}(k) + \widehat{w}_0(k)$, $\widehat{w}_0(k)$ small and $\langle \Phi', w_0 \rangle = 0$. Generate $\widehat{w}_n(k) = \hat{u}_n(k) - \widehat{\Phi}(k)$ by linearized operators:

$$\widehat{w}_{n+1}(k) = \gamma m_n \widehat{\Phi}(k) + p \frac{\widehat{\Phi^{p-1}} \star \widehat{w}_n(k)}{c + v(k)} \tag{3.1}$$

$$m_n = (1-p) \frac{\int_{-\infty}^{\infty} \widehat{\Phi^p}(k) \widehat{w}_n(k) dk}{\int_{-\infty}^{\infty} \widehat{\Phi^p}(k) \widehat{\Phi}(k) dk} = M_n - 1 \tag{3.2}$$

proof: calculation, done in handout.

I Proof of Proposition 3.1

Define space $X_p := \{U \in L^2 : \langle \Phi^p, U \rangle = 0\}$.
We decompose $\widehat{w}_n(k) = \hat{u}_n(k) - \widehat{\Phi}(k)$ into

$$w_n = a_n \Phi(x) + q_n(x) \quad , \quad q_n(x) \in X_p \tag{3.3}$$

Immediately (3.2, 3.3): $m_n = (1-p)a_n$ and by short calculations (see handout):

$$m_{n+1} = [p - \gamma(p-1)] m_n \tag{3.4}$$

$$q_{n+1}(x) = q_n(x) - (c + \mathcal{L})^{-1} \mathcal{H}\, q_n(x) \tag{3.5}$$

Want to prove that $w_n \overset{n\to\infty}{\longrightarrow} 0$ to conclude that spectral radius of (3.1), (3.2) less than 1.

(1) $m_n \longrightarrow 0$ if $1 < \gamma < \frac{p+1}{p-1}$. Superlinear: $\gamma = \frac{p}{p-1}$.

II Proof of Proposition 3.1

(2) $q_n \longrightarrow 0$:
Decompose q_n into EF of $(c+\mathcal{L})^{-1}\mathcal{H}$ (see later) in X_p. We need two Lemmata (proven later):

Lemma 2.4
$\sigma\left((c+\mathcal{L})^{-1}\mathcal{H}\right)$ in $X_p(\mathbb{R})$ has $n(\mathcal{H})-1$ negative EV.

Lemma 2.5
Positive spectrum of $(c+\mathcal{L})^{-1}\mathcal{H}$ in $X_p(\mathbb{R})$:

1. Infinitely many discrete EV. $0<\lambda<1$ (accumulating to 1^-).
2. If $\forall x\in\mathbb{R}$: $\Phi^{p-1}(x)\geq 0$: no EV. >1.
3. If $\exists x_0\in\mathbb{R}:\Phi^{p-1}(x_0)<0$, we also have infinitely many discrete EV. in $1<\lambda<\lambda_{\max}$ (accumulating to 1^+), and $\lambda_{\max}<1+\frac{p}{c}\mid\min_{x\in\mathbb{R}}\Phi^{p-1}(x)\mid<\infty$.

III Proof of Proposition 3.1

We had $q_{n+1}(x) = q_n(x) - (c+\mathcal{L})^{-1}\,\mathcal{H}\,q_n(x)$ (3.5)

Φ' is EF of $(c+\mathcal{L})^{-1}\,\mathcal{H}$ to EV 0, but $\langle \Phi', q_0\rangle = 0$ ($\langle w_0, \Phi'\rangle = 0$, use $\langle \Phi, \Phi'\rangle = 0$) implies $\langle \Phi', q_n\rangle = 0$ by induction (use 3.5).

$$q_n(x) = \sum_{k=1}^{n(\mathcal{H})-1} \alpha_k^{(n)} U_k(x) + \sum_{0<\lambda_k<1} \beta_k^{(n)} U_k(x) + \sum_{1<\lambda_k\le\lambda_{\max}} \gamma_k^{(n)} U_k(x) \quad (3.6)$$

$$\alpha_k^{(n+1)} = (1+|\lambda_k|)\alpha_k^{(n)} \qquad \lambda_k < 0 \qquad (3.7)$$

$$\beta_k^{(n+1)} = (1-\lambda_k)\beta_k^{(n)} \qquad 0 < \lambda_k < 1 \qquad (3.8)$$

$$\gamma_k^{(n+1)} = (1-\lambda_k)\gamma_k^{(n)} \qquad 1 < \lambda_k \le \lambda_{\max} \qquad (3.9)$$

For (max. linear !) convergence to 0 we need $n(\mathcal{H}) = 1$ and assumption 2.7.

IV Proof of Proposition 3.1

Remark: Add $\sum_{\lambda_j=0} \delta_j^{(n)} U_0^j(x)$ to q_n:

- If w_0 not orthogonal to $\Phi' \longrightarrow$ Iteration of w_n converges to $c_0\Phi'$. translation in x of $\Phi(x)$ to $\Phi(x + c_0)$, since we have linearized operator (first order correction !).
- $\text{Ker}(\mathcal{H}) > 1$, non-orthogonal w_0: Not necessarily convergence to Φ', bifurcation. We need assumption 2.1.

q.e.d Proposition 3.1

Outline

Preliminaries

Orthogonal Basis

$(c+\mathcal{L})^{-1}\mathcal{H}$ in L^2: $\mathcal{H}$ selfadjoint, $(c+\mathcal{L})$ positive $\longrightarrow$ EF of generalized EVP (2.4) $\mathcal{H}\,U = \lambda(c+\mathcal{L})U$ form an orthogonal basis of L^2.

Lagrange Multipliers

Analysis II: Extremum of function $f(x,y)$ under constraint $\phi(x,y)=0$ computed through 3 equations:

$$\phi(x,y)=0 \quad \nabla\left[f(x,y)+\lambda\phi(x,y)\right]=0$$

Generalize to infinite dimensions: looking for extremum of $F[\psi]$ under constraint $C[\psi]=0$:

$$C[\psi]=0 \quad \frac{\delta}{\delta\psi}\left(F[\psi]+\nu C[\psi]\right)=0$$

Lemma 2.3

Lemma 2.3

The negative space of $\mathcal{H}$ in $X_p(\mathbb{R})$ has dimension $n(\mathcal{H}) - 1$.

Proof:

Need to find solutions (μ, ψ) to $(\mathcal{H} - \mu)\psi = 0$ under constraint that $\langle \Phi^p, \psi \rangle = 0$. Use Lagrange Multiplier ν to get

$$\langle \Phi^p, \psi \rangle = 0 \quad \frac{\delta}{\delta \psi}\left(\frac{1}{2}\langle (\mathcal{H} - \mu)\psi, \psi \rangle + \nu \langle \Phi^p, \psi \rangle\right) = 0$$

$$\text{in other words: } \langle \Phi^p, \psi \rangle = 0 \quad \mathcal{H}\psi = \mu\psi - \nu\Phi^p(x) \tag{2.7}$$

Decompose ψ with L^2 EV-EF pairs (μ_k, u_k), $\mu \notin \sigma_{X_p}(\mathcal{H})$:

$$\psi(x) = \nu \left[\sum_{\mu_k < 0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) + \sum_{\mu_k > 0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) \right] \tag{2.8}$$

$$\psi(x) = \nu \left[\sum_{\mu_k<0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) + \sum_{\mu_k>0} \frac{\langle u_k, \Phi^p \rangle}{\mu - \mu_k} u_k(x) \right]$$

1. $u_k \in X_p$: μ_k is eigenvalue of $\mathcal{H}$ over X_p.
2. $u_k \notin X_p$: Still need to fullfill constraint equation:

$$F(\mu) = \frac{1}{\nu} \langle \Phi^p, \psi \rangle = \sum_{\mu_k<0} \frac{|\langle \Phi^p, u_k \rangle|^2}{\mu - \mu_k} + \sum_{\mu_k>0} \frac{|\langle \Phi^p, u_k \rangle|^2}{\mu - \mu_k} \overset{!}{=} 0 \qquad (2.9)$$

$$n_{X_p}(\mathcal{H}) = \#(1) + \#(2).$$

Discussion of (2.9):

- Mon. decr. for $\mu \leq 0$ and $\mu \neq \mu_k$, cont. in (μ_{k-1}, μ_k).
- Eigenvalues μ_k of (1): F continuous at $\mu = \mu_k$.
- $F \overset{\mu \to -\infty}{\longrightarrow} 0^-$
- $F(0) = -\langle \Phi^p, \mathcal{H}^{-1} \Phi^p \rangle = \frac{1}{p-1} \langle \Phi^p, \Phi \rangle > 0$
- $\pm\infty$ at $\mu = \mu_k$ for $u_k \notin X_p$.

Have $\#(2) = \#\text{poles} - 1$. Get $n(\mathcal{H}) - 1$ negative EV over X_p.

q.e.d. Lemma 2.3

Lemma 2.4

Lemma 2.4

The spectrum of $(c + \mathcal{L})^{-1}\,\mathcal{H}$ in $X_p(\mathbb{R})$ has $n(\mathcal{H}) - 1$ negative eigenvalues λ.

Proof:

$n(\mathcal{H}) =$ dimension of negative space of quadratic form $\langle U, \mathcal{H}\,U\rangle$ $\equiv n\left(\langle U, \mathcal{H}\,U\rangle\right)$, $U \in X_p(\mathbb{R})$.

By generalized inertial theorem $n\left(\langle U, \mathcal{H}\,U\rangle\right)$ is the same in any orth. basis of X_p diagonalizing $\langle U, \mathcal{H}\,U\rangle$ wrt. positively weighted inner product:

- Orth. (wrt. $\langle ., .\rangle$) basis through $\psi(x)$ as defined in (2.8).
- Orth. (wrt. $\langle (c + \mathcal{L}) ., .\rangle$) basis out of generalized EVP (2.4).

q.e.d. Lemma 2.4

Lemma 2.5

Lemma 2.5

Positive spectrum of $(c+\mathcal{L})^{-1}\mathcal{H}$ in $X_p(\mathbb{R})$:

1. Infinitely many discrete EV. $0 < \lambda < 1$ (accumulating to 1^-).
2. If $\forall x \in \mathbb{R}$: $\Phi^{p-1}(x) \geq 0$: no EV. > 1.
3. If $\exists x_0 \in \mathbb{R} : \Phi^{p-1}(x_0) < 0$, we also have infinitely many discrete EV. in $1 < \lambda < \lambda_{\max}$ (accumulating to 1^+), and $\lambda_{\max} < 1 + \frac{p}{c} \mid \min_{x\in\mathbb{R}} \Phi^{p-1}(x) \mid < \infty$.

Proof (bounds only):

Continuity / Discreteness of spectrum out of spectral theory.
Rewrite (2.4) as

$$(c+\mathcal{L})U - \frac{p}{1-\lambda}\Phi^{p-1}(x)U = 0 \qquad (2.12)$$

$$(c+\mathcal{L})U - \frac{p}{1-\lambda}\Phi^{p-1}(x)U = 0 \quad (2.12)$$

Multiply (2.12) by U and integrate:

$$\lambda = 1 - p\frac{\langle U, \Phi^{p-1}U\rangle}{\langle U, (c+\mathcal{L})U\rangle} \tag{2.13}$$

1. $\forall x \Phi^{p-1}(x) \geq 0 \longrightarrow \lambda < 1$.
2. $\exists x_0 : \Phi^{p-1}(x_0) < 0$:

$$\begin{aligned}\lambda &= 1 - p\frac{\langle U, \Phi^{p-1}U\rangle}{\langle U, (c+\mathcal{L})U\rangle} < 1 + p\frac{\left|\min\limits_{x\in\mathbb{R}}\Phi^{p-1}(x)\right|\langle U, U\rangle}{\langle U, (c+\mathcal{L})U\rangle} \\ &< 1 + p\frac{\left|\min\limits_{x\in\mathbb{R}}\Phi^{p-1}(x)\right|\langle U, U\rangle}{c\langle U, U\rangle} = 1 + \frac{p}{c}\left|\min\limits_{x\in\mathbb{R}}\Phi^{p-1}(x)\right|\end{aligned}$$

q.e.d. Lemma 2.5

Outline

Overview

Main Theorem 2.8

Let $\widehat{\Phi}(k)$ solution to (1.5), assumptions 1.1 (solution space) and 2.1 (Nullspace, bifurcation).
Petviashvili Iteration (1.8), (1.9) converges to $\widehat{\Phi}(k)$ in (small) neighbourhood (Continuity of linearized operator, Fixed Point Theorem) of $\widehat{\Phi}(k)$ if:

1. $1 < \gamma < \frac{p+1}{p-1}$ (Proposition 3.1, convergence of m_n)
2. $n(\mathcal{H}) = 1$ (Proposition 3.1, convergence of q_n)
3. assumption 2.7 is met. (Proposition 3.1, convergence of q_n)

"If any of the conditions are not met, the Petviashvili iteration diverges from $\widehat{\Phi}(k)$". (Bifurcation)

Remark: Generalization to more dimensions possible !

References

[PS] Dmitry E. Pelinovsky, Yury A. Stepanyants, *Convergence of Petviashvili's Iteration Method for Numerical Approximation of Stationary Solutions of Nonlinear Wave Equations*, SIAM J. NUMER. ANAL., Vol. 42, No. 3, pp. 1110-1127.

[HP] V. Huson, J.S. Pym, *Applications of Functional Analysis and Operator Theory*, Mathematics in Science and Engineering, Volume 146, Academic Press, 1980.