

# Notes on The Sine Gordon Equation

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## **Abstract**

In this seminar, we will introduce the Sine-Gordon equation, and solve it using a Baecklund transformation. Furthermore, we also give a numeric solution using a split-step algorithm, and also present two physical applications of the Sine-Gordon equation.

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# 1 Introduction

First of all, we want to mention why the Sine-Gordon equation (SGE) is important to study. From the model building perspective, there are various interesting examples making use of the SGE, such as the propagation of splay waves on a lipid membrane, one dimensional models for elementary particles, self-induced transparency of short optical pulses and domain walls in ferroelectric and ferromagnetic materials.

In this talk, we will look in more detail on the propagation of crystal defects, as predicted by the Frenkel Kontorova model, and on the propagation of fluxons in a long Josephson junction. Both topics are outstanding examples of the success of solid state physics throughout the last century, where the Frenkel-Kontorova model was found in the forties, and the huge success of devices using Josephson junctions has been a result of the discovery of high-temperature superconductors in the eighties.

The second point worth noting in the introduction is the historical development of the SGE. It seemingly first appeared in differential geometry, where it was used to describe surfaces with a constant negative Gaussian curvature. In that context, it was the first equation to be solved with a Bäcklund transformation, which was later used for a variety of equations. From a more physical perspective, at the birth of quantum mechanics, the Klein-Gordon equation was found when trying to find a relativistically invariant equation. Due to the Klein-Gordon equation's enormous popularity, the SGE was named so as a wordplay.

## 2 The Sine-Gordon equation

### 2.1 Deriving the SGE

We introduced solitons as the solutions to a nonlinear (wave) equation, where the nonlinearity and the dispersion balance each other out, so that there exists a stable but nontrivial solution. The way how one usually obtains such field equations in physics is by using the principle of least action, which states that the path a physical system takes, is the one that minimizes the action:  $\Delta S = 0$ . The action for a field

theory is usually defined by

$$\int \mathcal{L} d^4x = \int \underbrace{\int \mathcal{L} dx}_{=L} dt = S \quad (1)$$

Using the principle of least action, and (1), one can obtain the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0 \quad (2)$$

Suppose we have a given Lagrangian density, then we can plug it into the Euler-Lagrange equations to obtain the equations of motion. One of the classical Lagrangians where one does this, is

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - U(\phi) \quad (3)$$

It is easy to verify that this Lagrangian density's corresponding equations of motion are

$$\partial_\mu \partial^\mu \phi + \left( \frac{\partial U}{\partial \phi} \right) = 0 \quad (4)$$

For the SGE, we now just take  $U = \cos(\phi)$  or  $U = 1 - \cos(\phi)$ , which yield identical equations of motion, namely

$$\partial_\mu \partial^\mu \phi + \sin(\phi) = 0 \quad (5)$$

If we restrict ourselves to one space and one time dimension, it is

$$\frac{\partial^2}{\partial t^2} \phi - \frac{\partial^2}{\partial x^2} \phi + \sin(\phi) = 0 \quad (6)$$

As a side remark, we note both the Klein-Gordon equation's potential and the  $\phi^4$ -potential:

$$U(\phi) = m^2 \phi^2 \quad \text{or} \quad U(\phi) = (m^2 - \phi^2)^2 \quad (7)$$

Obviously, the Klein Gordon potential has a minimum at  $\phi = 0$ . The  $\phi^4$ -potential on the other hand, has a minimum at  $\phi = m$  which corresponds to a non-zero vacuum expectation value. In the case where we have an interaction term in the Lagrangian, i.e., we look at the interaction of  $\phi$  with another (probably) nonscalar field, the shift of our scalar field then invokes a "mass" term inside the interaction term. This is

commonly known as the Higgs mechanism.

Note however, that there are many choices possible for such a potential. Interestingly enough, our mentioned  $U(\phi) = (m^2 - \phi^2)^2$  potential also has solitonic solutions. The existence of such kinks has inspired some physicists to propose brane fixing mechanisms in rather exotic extra dimension theories. That is, to provide an answer to the "why do we live only in a 4 dimensional subspace of the world"-question.

## 2.2 The General Idea of the Baecklund Transformation

The basic idea of the Baecklund transformation is that if we have a solution, even if it is a trivial one, there exists a transformation which transforms this solution into a new solution. To be more precise: Suppose we want to solve an equation  $P(u) = 0$  where  $P$  is a (nonlinear) differential operator and  $u$  is a space and time dependent function. Now we take another PDE  $Q(v) = 0$  where the  $Q$  is again a (nonlinear) differential operator for the new space and time dependent function  $v$ . Then we can find two relations between the variables  $u$  and  $v$

$$R_1(u, v, u_x, v_x, u_t, v_t, \dots; x, t) = 0 \quad (8)$$

$$R_2(u, v, u_x, v_x, u_t, v_t, \dots; x, t) = 0 \quad (9)$$

We now say that  $R_i = 0$  is a Baecklund transformation if the relations

1. are integrable for  $v$  when  $P(u) = 0$
2. then the resulting  $v$  must be a solution of  $Q(v) = 0$
3. and vice versa, it must be true too.

Additionally, we say that  $R_i$  are auto-Baecklund transformations, if  $P = Q$  such that both  $u$  and  $v$  satisfy the same equation.

### 2.2.1 Example: Laplace equation

Let us solve the Laplace equation in two dimensions using a Baecklund transformation.

$$\Delta u = (\partial_x^2 + \partial_y^2)u = 0 \quad (10)$$

We now look for an auto-Baecklund transformation, that is we take the following initial Operators

$$P(u) = \Delta u = (\partial_x^2 + \partial_y^2)u = 0, \quad \text{and} \quad Q(v) = \Delta v = (\partial_x^2 + \partial_y^2)v = 0. \quad (11)$$

The transformations we look for are thus:

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v \quad (12)$$

By taking the cross-derivatives

$$\partial_x(u_x) = \partial_x(v_y) \quad \text{and} \quad \partial_y(u_y) = \partial_y(-v_x) \quad (13)$$

we get the auto-Baecklund transformations:

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_y u = -\partial_x v$$

note that by taking the cross derivatives  $\partial_x(u_x) = \partial_x(v_y)$  and  $\partial_y(u_y) = \partial_y(-v_x)$  and by adding the two equations, the original laplace equation comes out. Now we take the trivial solution  $v(x, y) = xy$  and plug it into the Baecklund transformation, to obtain  $u_x = x$  and  $u_y = -y$ . Thus we get  $u = \frac{1}{2}(x^2 - y^2)$ .

## 2.3 the Baecklund Transformation for the SGE

### SGE rewritten

First we need to rewrite the SGE by using other differential operators

$$\partial_\xi = (\partial_x + \partial_t) \quad \text{and} \quad \partial_\zeta = (\partial_x - \partial_t) \quad (14)$$

For these operators, the rewritten SGE becomes

$$\partial_\xi \partial_\zeta u = \sin u \quad (15)$$

### verifying the validity of our transform

Here we take the following pair of equations for the  $R_i$ .

$$\frac{1}{2}(u+v)_\xi = a \sin\left(\frac{u-v}{2}\right) \quad \text{and} \quad \frac{1}{2}(u-v)_\zeta = \frac{1}{a} \sin\left(\frac{u+v}{2}\right) \quad (16)$$

Furthermore we show that this pair of equations is a valid Baecklund transform. For simplicity, we will refer to the transformed variables  $\zeta$  and  $\xi$  again as  $t$  and  $x$ . Take the cross-derivatives and get

$$\frac{1}{2}(u+v)_{xt} = \frac{a}{2}(u-v)_t \cos\left(\frac{u-v}{2}\right) = \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right) \quad (17)$$

and

$$\frac{1}{2}(u-v)_{tx} = \frac{1}{2a}(u+v)_x \cos\left(\frac{u+v}{2}\right) = \sin\left(\frac{u-v}{2}\right) \cos\left(\frac{u+v}{2}\right) \quad (18)$$

Thus the  $P$  and  $Q$  equations are

$$u_{xt} = \sin(u) \quad \text{and} \quad v_{xt} = \sin(v) \quad (19)$$

### obtaining the new solution

We begin by taking  $v = 0$  as a trivial solution. Then the Baecklund transformation (16) yields

$$u_x = 2a \sin\left(\frac{1}{2}u\right) \quad \text{and} \quad u_t = \left(\frac{2}{a}\right) \sin\left(\frac{1}{2}u\right) \quad (20)$$

Integrating the two equations yields

$$2ax = \int^u \frac{du}{\sin\left(\frac{1}{2}u\right)} = 2 \log \left| \tan\left(\frac{1}{4}u\right) \right| + f(t) \quad (21)$$

and

$$2\left(\frac{t}{a}\right) = \int^u \frac{du}{\sin\left(\frac{1}{2}u\right)} = 2 \log \left| \tan\left(\frac{1}{4}u\right) \right| + g(x) \quad (22)$$

then taking the exponential on both sides

$$\tan\left(\frac{1}{4}u\right) = C \exp(ax + t/a) \quad (23)$$

which is the same as

$$u(x, t) = 4 \arctan(C \exp(ax + t/a)) \quad (24)$$

## 2.4 Miscellaneous Remarks

### 2.4.1 Lorentz Invariance

A remarkable feature of the SGE, which makes it very useful for physics, is its invariance under Lorentz transformations. Note that with  $\gamma = 1/(1-v^2)^{1/2}$  (where we set  $c = 1$ )

$$x' = \gamma(x - vt) \quad \text{and} \quad t' = \gamma(t - vx) \quad (25)$$

then a wave equation must be Lorentz invariant, respectively the d'Alembert operator must be equal to its Lorentz transformed version,  $\square = \square'$ .

$$\partial_x^2 u - \partial_t^2 u = \partial_{x'}^2 u - \partial_{t'}^2 u$$

This can be shown by transforming the derivatives (note that  $x'$  is a function of two variables, thus taking the derivative with respect to two variables :

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \underbrace{\frac{\partial x'}{\partial x}}_{=\gamma} + \frac{\partial}{\partial t'} \underbrace{\frac{\partial t'}{\partial x}}_{=-\gamma v} \quad (26)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'} \underbrace{\frac{\partial x'}{\partial t}}_{=-\gamma v} + \frac{\partial}{\partial t'} \underbrace{\frac{\partial t'}{\partial t}}_{=\gamma} \quad (27)$$

then

$$\partial_x^2 = \gamma^2 (\partial_{x'}^2 + v^2 \partial_{t'}^2 - 2v \partial_{x'} \partial_{t'}) \quad (28)$$

$$\partial_t^2 = \gamma^2 (\partial_{t'}^2 + v^2 \partial_{x'}^2 - 2v \partial_{x'} \partial_{t'}) \quad (29)$$

then

$$\partial_x^2 - \partial_t^2 = \gamma^2 \left( \underbrace{(1 - v^2)}_{=\gamma^{-2}} \partial_{x'}^2 - \underbrace{(1 - v^2)}_{=\gamma^{-2}} \partial_{t'}^2 + 2v \partial_{x'} \partial_{t'} - 2v \partial_{x'} \partial_{t'} \right) \quad (30)$$

$$= \partial_{x'}^2 - \partial_{t'}^2 \quad (31)$$

So we have shown the invariance of the d'Alembert operator under Lorentz transformations. As a Lorentz transformed sinus remains a sinus, the equation still remains a Sine-Gordon equation, thus it is Lorentz invariant.

## 2.4.2 Other Analytic Solutions

We denote the soliton velocity by  $\lambda$ , as in [1]. For completeness, without proof, we want to mention some other analytic solutions of the SGE. For an easy verification



of these solutions, use the following hints: all analytic solutions shown here are of the form  $u = 4 \arctan \varphi$  From analysis, one has

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad 2 \arcsin x = \arcsin(2x\sqrt{1-x^2}) \quad (32)$$

after a few lines of calculations, one obtains :

$$\sin(4 \arctan x) = 4x \frac{(1-x^2)}{(1+x^2)^2} \quad (33)$$

For the differential part, note that:

$$\partial_x^2 u = \frac{\varphi'' - \varphi' 2\varphi\varphi' + \varphi''\varphi^2}{(1+\varphi^2)^2} \quad (34)$$

The other solutions are then:

- Breather

$$u(x, t) = 4 \arctan \left( \frac{(1-\lambda^2)^{1/2}}{\lambda} \frac{\sin(\lambda(t-t_0))}{\cosh\left((1-\lambda^2)^{1/2}(x-x_0)\right)} \right)$$

- Kink-Antikink solution

$$u(x, t) = 4 \arctan \left( \frac{\lambda \cosh\left(x/(1-\lambda^2)^{1/2}\right)}{\sinh\left(\lambda t/(1-\lambda^2)^{1/2}\right)} \right)$$

## 3 Applications

### 3.1 Frenkel-Kontorova Soliton

#### 3.1.1 Inside Solid-State Physics

The Frenkel Kontorova model is one way to describe in solid state physics the dynamic behaviour of defects. We briefly want to mention why the study of defects is important to the understanding.

The basic approach to solid state physics is that we have both perfect periodicity and an infinite lattice. Together with the fact that (quantum mechanically) interacting energy levels split, we can see how a band structure forms itself. Another basic approach that is usually made is that all atoms interact within a harmonic approximation. Within this approximation, one introduces usually phonons, that is lattice vibrations, and other quantities. Although in some sense the phonons can be seen as a three dimensional defect, because they deform the whole lattice, it appears very much the same as the perfect lattice, due to the small deviations. Both of these properties depend to some extent on the fact that a real solid is made of about  $10^{23}$  particles. For such a high number, it is easy to see why a band becomes a "continuous" structure and why the interaction between two particles can be assumed to be harmonic.

by contrast to these ideas we have the fact that there not only exists a variety of defects, but they can change physical properties enormously, even in small numbers. Examples include point defects, which are both unavoidable for thermodynamic reasons, as well as they are sometimes desired too, for example impurities in semiconductors can lead to different electric properties, as it is used in chips, and impurities are also responsible for the color of some crystals, such as ruby or sapphire. Technical usage of these impurities has been known for a very long time. For example take steel where these foreign atoms (carbon atoms) lead to the pinning of one dimensional defects, which are responsible for many elastic properties, i.e. the movement of dislocations becomes more difficult, which leads to a harder material.

Here we will look at another defect with solitonic properties: The Frenkel-Kontorova soliton which describes a movement of displaced atoms. For that, we will start by repeating some facts about pendula.

### 3.1.2 discrete example for SGE

As the harmonic oscillator has a potential energy of  $\frac{\omega_0^2}{2}x^2$ , this gives the commonly known equations of motion for a harmonic oscillator:

$$\frac{d^2}{dt^2}x + \omega_0^2x = 0 \tag{35}$$

which has the "usual" sinusoid solutions. This is usually used to describe a pendulum with a small amplitude. However, for large excitations of a pendulum ( $\omega_0 = \sqrt{\frac{g}{l}}$ ,

instead the potential becomes proportional to  $(1 - \cos x)$  with the equations of motion:

$$\frac{d^2x}{dt^2} + \omega_0^2 \sin x = 0 \quad (36)$$

Which is obviously a nonlinear differential equation. We are now interested in what happens when we couple such pendula and let them interact, and then we will take the continuous limit. For that, consider first the following potential of the substrate as externally given:

$$V(u_n) = V_0 \left( 1 - \cos \left( \frac{2\pi u_n}{a} \right) \right) \quad (37)$$

Furthermore, we assume that the interaction energy term does not contain higher order interactions, i.e. it is harmonic.

$$W(u_{n-1}, u_n) = C/2(u_n - u_{n-1})^2 \quad (38)$$

Then the total Hamiltonian is

$$H = \sum_n \frac{p_n^2}{2m} + W(u_{n-1}, u_n) + V(u_n) \quad (39)$$

### 3.1.3 continuous limit

The equations of motion belonging to the total Hamiltonian (39) are then:

$$m \frac{d^2 u_n}{dt^2} = C(u_{n+1} + u_{n-1} - 2u_n) - \frac{2\pi V_0}{a} \sin \frac{2\pi u_n}{a}$$

We can see the analogy between these equations of motion and the SGE by approximating

$$C(u_{n+1} + u_{n-1} - 2u_n) = C a^2 \frac{(u_{n+1} - u_n)/a - (u_n - u_{n-1})/a}{a} \approx C a^2 \frac{d^2 u}{dx^2}$$

Then obviously, the SGE holds.

As for the meaning of the SGE in solid systems, we can think of the following: assume a perfect lattice which generates a periodic potential, i.e. the periodic potential is made up from the potentials of the single atoms. For simplicity, we can assume the periodic potential to be a sine, which turns out to be a good approximation for

most cases. Then we assume that a misplaced atom still feels an undisturbed periodic potential, which we justify by the far-range influence of an infinite amount of perfectly ordered atoms. If we now reduce our solid state system to one dimension, then we know that there are solitons existing. That also means that not all atoms are in the minimum of the potential for all times. Since we know that these solitonic solutions move, we also have a mechanism for the movement of displacements, which permits the understanding of plastic deformation of bodies.

However, note that because the atomic lattice is not really a continuous system, these solitons are just an approximation, which is a very good one again due to the fact that we have  $10^{23}$  particles in the lattice, which is by far a bigger system than the 257 "particles" used in our computer simulation. Of course, even such a small system already gives a good validity of the Sine-Gordon kink. This comparison perhaps also shows one of the difficulties of computational solid-state physics: in some sense we are limited to effects of small systems, as we do not have the computing power to "really" simulate a system with  $10^{23}$  particles.

Note also that often dislocations are introduced with elasticity theory, where the beginning assumption is a continuous system, whereas we started the other way around, namely by a discrete system, which then is grown to an (almost) continuous one. Interesting however is also the following comparison: when "classical" (one-dimensional) dislocations and other similar defects move inside the body, they usually radiate phonons, however our solitonic solutions do not radiate phonons.

## 3.2 Josephson Junction

### 3.2.1 Purpose

The SGE does also appear in a totally different context within modern solid state physics. Namely, within elongated Josephson junctions. Before treating elongated Josephson junctions, we will say somewhat more about their purpose and about "ideal" Josephson junctions.

First, some words about the Josephson effect: In a setup that looks like a "sandwich" of superconductors with a thin oxide layer or a bottleneck in between, such that only single electrons and cooper pairs, i.e. paired electrons, can tunnel through that layer, there is an amazing effect: even without applying an outer electric field, there will be a measurable direct current through the junction. When we apply a voltage between

the two superconductors, we will get an alternating current through the junction, which is what we will prove in the first part of this chapter.

In a so called superconducting quantum interferometry device, two such josephson junctions are then used to measure interferences between two paths, thus very small magnetic fields can be detected. These devices are used then for example in brain science and in many other areas.

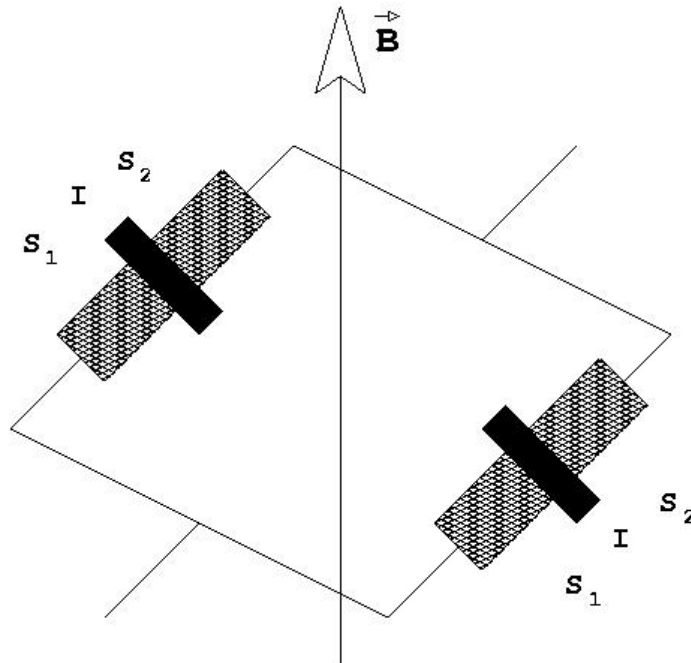


Figure 1: Note that in a real squid, the branching points might be made of superconductors too

### 3.2.2 AC Josephson effect

Here will mainly follow the derivation in [3], which is similar to the one presented in [2]. The AC Josephson effect which we will look at here, provides then the non-linear element, i.e. the sine part in the (almost) SGE of the elongated Josephson junction.

For the system made of these two weakly coupled superconductors, the following version of Schroedingers equation is valid, where  $\Phi =$  is the "many-particle-

wavefunction",  $\mathcal{H}_i$  the total Hamilton operator for  $S_i$  and  $T$  is the tunnel-coupling

$$i\hbar\partial_t\Phi_1 = \mathcal{H}_1\Phi_1 + T\Phi_2 \quad (40)$$

$$i\hbar\partial_t\Phi_2 = \mathcal{H}_2\Phi_2 + T\Phi_1 \quad (41)$$

If we now apply a voltage, and choose the origin symmetric, and also take  $\Phi_1$  as an almost eigensolution with the energy eigenvalue  $qU$ , our equations can be rewritten.

$$i\hbar\partial_t\Phi_1 = \frac{qU}{2}\Phi_1 + T\Phi_2 \quad (42)$$

$$i\hbar\partial_t\Phi_2 = \frac{qU}{2}\Phi_2 + T\Phi_1 \quad (43)$$

With  $|\Phi|^2 = n_s/2 = n_c$  as the density of cooper pairs, we rewrite our many particle wavefunction

$$\Phi_1 = \sqrt{n_{c1}}e^{i\phi_1} \quad \text{and} \quad \Phi_2 = \sqrt{n_{c2}}e^{i\phi_2}$$

then plugging in  $\Phi_i$  in the equation (42) gives with separation of real and imaginary parts

$$\hbar/2\partial_t n_{c1} \sin \phi_1 - (\hbar\partial_t \phi_1 + qU/2) n_{c1} \cos \phi_1 = T\sqrt{n_{c1}n_{c2}} \cos \phi_2$$

$$\hbar/2\partial_t n_{c1} \cos \phi_1 - (\hbar\partial_t \phi_1 + qU/2) n_{c1} \sin \phi_1 = T\sqrt{n_{c1}n_{c2}} \sin \phi_2$$

of course, there are analogous equations for the other equation (43)

Now we can multiply both equations with  $\cos \phi_1 \sin \phi_1$  and the other way around and then subtract from each equation the other equation to obtain a new set, together with the same derivation for (43).

$$j_{T\alpha} \quad \partial_t n_{c1} = \frac{2}{\hbar}T\sqrt{n_{c1}n_{c2}} \sin(\phi_2 - \phi_1) \quad (44)$$

$$j_{T\alpha} \quad \partial_t n_{c2} = \frac{-2}{\hbar}T\sqrt{n_{c1}n_{c2}} \sin(\phi_2 - \phi_1) \quad (45)$$

$$\partial_t \phi_1 = \frac{1}{\hbar}T\sqrt{\frac{n_{c2}}{n_{c1}}} \cos(\phi_2 - \phi_1) - \frac{qU}{2\hbar} \quad (46)$$

$$\partial_t \phi_2 = \frac{1}{\hbar}T\sqrt{\frac{n_{c1}}{n_{c2}}} \cos(\phi_2 - \phi_1) + \frac{qU}{2\hbar} \quad (47)$$

What we get out is a proportionality for the tunneling current. This set can now be simplified for identical superconductors ( $n_{c1} = n_{c2} = n_c$ ) to give the amazing result that there is an alternating current through the Josephson junction, even though we applied a constant voltage between the two superconductors. For the following we will use the phase difference  $\theta$

$$\partial_t n_{c1} = (2T/\hbar)n_c \underbrace{\sin(\phi_2 - \phi_1)}_{:=\theta} = -\partial_t n_{c2} = j_T \quad (48)$$

$$\hbar(\partial_t \phi_2 - \partial_t \phi_1) = -qU \quad (49)$$

### 3.2.3 Elongated Josephson Junction

First note the homogeneous Maxwell's equations and that  $E = \frac{\partial U}{\partial x} = \frac{qU}{d}$  in one dimension

$$\nabla \times E = -\partial_t B \quad (50)$$

$$\nabla \times B = \mu_0 j + \mu_0 \epsilon \partial_t E \quad (51)$$

without further explanation, note that the following ODE is valid for an ideal "resistively and capacitively shunted junction" for the external current

$$j_B = j_t + \frac{U}{RS} + \frac{C}{S} \frac{dU}{dt} \quad (52)$$

where C is the capacity of the junction, R the resistivity, U the voltage, and S an area. In this limit, for the Josephson junction this equation becomes

$$\frac{d^2}{dt^2} \theta + \frac{1}{RC} \frac{d\theta}{dt} + \omega_0^2 \sin \theta = \frac{2eS}{C\hbar} j_B \quad (53)$$

which behaves like the equation for a disturbed excited pendulum

if we now insert (49) into the first of Maxwell's equations, we get

$$-\partial_x E_z = -\partial_x (U(x, t)/d) = -\partial_t B_y \quad (54)$$

$$\Leftrightarrow \partial_x \left( \frac{\hbar}{2ed} \partial_t \theta \right) = \partial_t B_y \quad (55)$$

by integrating over time, we see

$$B_y = \frac{\hbar}{2ed} \partial_x \theta + B_{0y} \quad (56)$$

We can now insert this expression for the magnetic field in  $y$ -direction into the other Maxwell equation.

$$u_z \nabla \times B = \mu_0 j + \mu_0 \epsilon \partial_t E = \frac{\hbar}{2ed} \partial_x^2 \theta + \mu_0 j_B \quad (57)$$

inserting this last term into the junction equation instead of only  $\mu_0 j_B$  gives

$$\theta_{tt} - c_0^2 \theta_{xx} + \underbrace{\frac{1}{RC} \theta_t + \omega_0^2 \sin \theta}_A = \underbrace{\frac{2eS}{C\hbar} j_B}_B$$

which in some limit behaves like a SGE, provided the two terms  $A$  and  $B$  are small, which can experimentally be provided by designing the junction in the right way. One can through another calculation conclude that the solitons of the SGE provide fluxons, that is a quantized magnetic flux.

## 4 Numerical Simulation

### 4.1 Another Notion For The SGE

The numerical examples were calculated using a Matlab program with a split step Fourier algorithm. Here we explain how it was programmed. Because this is a second order in time differential equation, we need first to get rid of one time differential. More precisely, note how we split the Sine Gordon equation: First, define

$$v = \frac{d}{dt} u \quad (58)$$

Second, rewrite the sine Gordon equation as

$$-\begin{pmatrix} u \\ v \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}}_{:=L} \begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \sin(u) \end{pmatrix}}_{:=S} = 0, \quad (59)$$



where the small  $t$  signifies taking the derivative with respect to time. For the simulations, we will start with given initial conditions and look at the time evolution of the problem. The split-step time evolution we then use is the following:

$$\begin{pmatrix} u \\ v \end{pmatrix} (t + dt) = \exp\left(L\frac{dt}{2}\right) \exp(Sdt) \exp\left(L\frac{dt}{2}\right) \begin{pmatrix} u \\ v \end{pmatrix} (t) + \mathcal{O}(dt^3) \quad (60)$$

where  $e^{Sdt}$  is a formal notation for the solution operator of the part

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = S(u, v) \quad (61)$$

The linear step  $\exp\left(L\frac{dt}{2}\right)$  then can be solved using a fourier transform, as

$$\begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}^\wedge = \begin{pmatrix} 0 & 1 \\ -k^2 & 0 \end{pmatrix} \quad (62)$$

## 4.2 Pseudo-Code

The program thus works conceptually in the following way:

1. get initial conditions, that is a  $U_0 := \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$  - vector.  
 $\Rightarrow$  Start the main loop
2. do a Fourier transform of  $U_0$
3. get  $\widehat{U}_1 = \widehat{L}\widehat{U}_0$
4. inverse fourier transform of  $\widehat{U}_1$
5. perform nonlinear step using  $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ dt \sin(u_1) \end{pmatrix}$
6. do a Fourier transform of  $U_2$
7. get  $\widehat{U}_3 = \widehat{L}\widehat{U}_2$

8. inverse fourier transform of  $\widehat{U}_3$   
 $\Rightarrow$  get  $U(t + dt)$
9. return to step 2

In principle, step 2 can be performed before starting the loop and then step 8 can be omitted for efficiency reasons.

### 4.3 Programmed Examples

For the coded examples which you can find in the appendix, note that you need to change both the windows and the Length parameter of the simulation suitable to your problem, as you switch from one example to another. The code will store your data in a .avi-movieclip. The pictures have been made with the same code, but instead of `getframe`, use a matrix to which you store the according element `meshmatrix(f,:)=uu(:);` , and after the loop do `mesh(meshmatrix)`

As you can see in the appendix, there are many programmed examples, including a test with an analytic and a numeric solution, to see deviations from the analytic solution. An idea to improve the code would be to use twice as large vectors that include both position and speed, and for the equation take a sparse matrix.

## 5 Appendix: Original Matlab Code

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Starting area
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%axis parameters for the plot
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
xmin = 0 ;

```

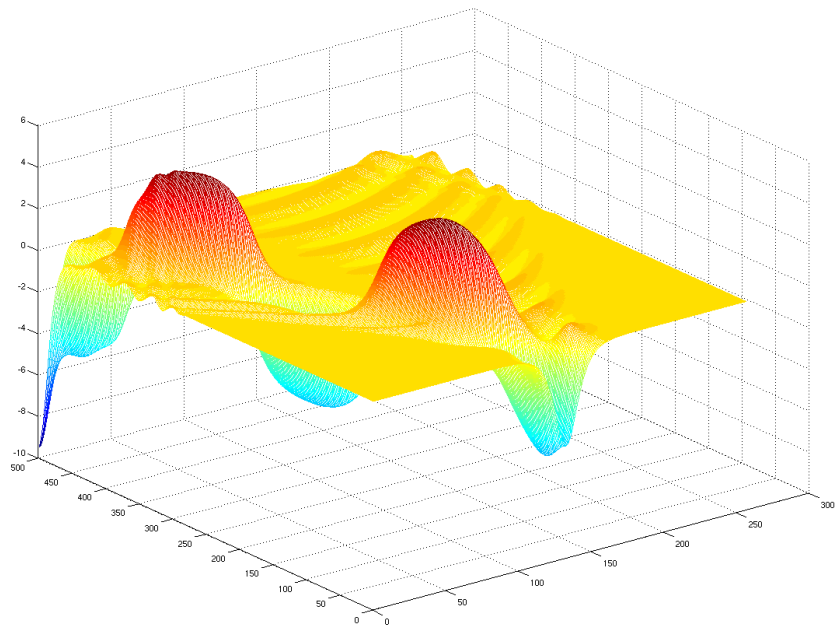


Figure 2: This image shows an imperfect moving breather

```
xmax = 2000 ;
ymin = -7 ;
ymax = 7 ;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%various starting parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
t=0.01;           %starting time for the propagating exact solution
lambda=0.992;    %velocity
const=1;         %other constant of integration for the solution
```

```
f=1;             %frameparameter for the movie
versetz=0;       %distance between analytic and numeric solution
```

```
xo=1             %spatial breather parameter
to=1             %time parameter for the breather
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Of Space
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
Nx = 257;        %amount of discrete points in the x-space;
L = 2000;        %total length of the interval
dx = L/(Nx-1);   %distance between two neighboring points
x = linspace(0,L,Nx); %position space
```

```
%note: due to better efficiency of fft algorithm, Nx should be chosen as
%2^n+1 , where the one additional point is needed for the doubling for the
%interval
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%And Time
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
Tfin = 1;
dt = 0.01;
Nt = floor(Tfin/dt);
```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%other parameters
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
qu=(1-lambda^2)^0.5;

```

```

phii=const*exp((x-lambda*t)/qu);

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% defining various vectors and matrices
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

F = zeros(2,Nx);
FF = zeros(2,Nx);

```

```

uu = zeros(1,Nx);
uv = zeros(1,Nx);

```

```

xx = zeros(1,Nx);
yy = zeros(1,Nx);

```

```

% the transforming vectors

```

```

uuu = zeros(1,2);
uuv = zeros(1,2);
uvv = zeros(1,2);

```

```

% the transforming matrices

```

```

%transmat = [1 1; i*k -i*k];
%eigvalmat = [ i*k 0 ; 0 -i*k];
%invmat = [1/2 1/(2*i*k); 1/2 -1/(2*i*k)];
%endmat=invmat*eigvalmat*transmat

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% initial vectors
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

F(1,:) = 4*atan(const*exp((x-L/2-lambda*t)/qu));
F(2,:) = -4*lambda*const*exp((x-L/2-lambda*t)/qu)./(qu*(1+const^2*exp(2*(x-L/2-lambda*t)/qu)));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% self generation of a soliton
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%F(1,:) = 4*(atan((x-L/2-lambda*t)/qu)+pi/2)/2;
%F(2,:) = -abs(lambda*const*(1/qu)./(qu*(1+const^2*(x-L/2-lambda*t).^2/qu)));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Soliton interactions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%a = linspace(0,L/2,Nx/2);
%b = linspace(L/2,L,Nx/2);

%AA = 4*atan(const*exp((-a+L/4-lambda*t)/qu));
%aaa = 4*lambda*const*exp((a-L/4-lambda*t)/qu)./(qu*(1+const^2*exp(2*(a-L/4-lambda*t)/qu)));

%BB = -4*atan(const*exp((b-L*3/4-lambda*t)/qu));
%bbb = -4*lambda*const*exp((b-L*3/4-lambda*t)/qu)./(qu*(1+const^2*exp(2*(b-L*3/4-lambda*t)/qu)));

%F(1,:)= [AA 0 BB];
%F(2,:)= [aaa 0 bbb];

%axis([xmin xmax ymin ymax])
%plot(x,F')
%pause(0.5)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% more complicated interactions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%a = linspace(0, L/4, Nx/4);
%b = linspace(L/4, L/2, Nx/4);
%c = linspace(L/2, L*3/4, Nx/4);
%d = linspace(L*3/4, L, Nx/4);

%AA = 4*atan(const*exp((a-L/8-lambda*t)/qu));
%aaa = -4*lambda*const*exp((a-L/8-lambda*t)/qu)./(qu*(1+const^2*exp(2*(a-L/8-lambda*t)/qu)));

%BB = 4*atan(const*exp((-b+L*3/8-lambda*t)/qu));
%bbb = 4*lambda*const*exp((b-L*3/8-lambda*t)/qu)./(qu*(1+const^2*exp(2*(b-L*3/8-lambda*t)/qu)));

```

```

%CC = -4*atan(const*exp((c-5*L/8-lambda*t)/qu));
%ccc = 4*lambda*const*exp((c-5*L/8-lambda*t)/qu)./(qu*(1+const^2*exp(2*(c-5*L/8-lambda*t)/qu)));

%DD = -4*atan(const*exp((-d+L*7/8-lambda*t)/qu));
%ddd = -4*lambda*const*exp((d-L*7/8-lambda*t)/qu)./(qu*(1+const^2*exp(2*(d-L*7/8-lambda*t)/qu)));

%fff=[AA BB CC 0 DD ];
%ffff=[aaa bbb 0 ccc ddd];
%F(1,:)= [AA BB 0 CC DD];
%F(2,:)= [aaa bbb 0 ccc ddd];

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%completely non-solitonic initial conditions
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

```

%a = linspace(0, L/4, Nx/4);
%b = linspace(L/4, L*3/4, Nx/2);
%c = linspace(L*3/4, L, Nx/4);

```

```

%F(1,:) = [a*0 0 4*sin(b) c*0];
%F(2,:) = [a*0 0 4*cos(b) c*0];

```

```

%or other variant

```

```

%a = linspace(0, L*9/40, Nx/4);
%b = linspace(L*9/40, L*31/40, Nx/2);
%c = linspace(L*31/40, L, Nx/4);

```

```

%F(1,:) = [a*0 0 6*exp(-(4*cos(b)).^2) c*0];
%F(2,:) = [a*0 0 6*sin(b).*cos(b).*exp(-(4*cos(b)).^2) c*0];

```

```

%instead: how a point excitation at the middle of the interval behaves
% like a chain of pendula

```

```

%F(1,int16(Nx/2))=4
%F(2,int16(Nx/2))=0

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```





```

% 0. getting vectors and their fourier transforms

for i=1:Nx
    uu(1,i)=F(1,i);
    uv(1,i)=F(2,i);
end

% Looking at the initial values and including in the movie
plot(x, F')
M(1)=getframe;
%M(2)=getframe;

xx = fft([uu uu(end-1:-1:2)],2*Nx-2);
yy = fft([uv uv(end-1:-1:2)],2*Nx-2);
qq = fftshift(xx);
qz = fftshift(yy);

% 1. Begin main loop

for r=2:Nt

    % 2. Piecewise construction of the first t/2 part

    for p=1:2*Nx-2
        uuu(1,1) = qq(1,p);
        uuu(1,2) = qz(1,p);

        k = (pi/L)*(-Nx+p);
        Mmat = [0 1; -k^2 0];

        uuu = expm(Mmat*dt/2)*transpose(uuu);

        qq(1,p) = uuu(1,1);
        qz(1,p) = uuu(2,1);
    end
end

```

```

% 3. Nonlinear part of the SGE :

% backtransforming into real space
xx = ifftshift(qq);
yy = ifftshift(qz);
xx = ifft(xx, 2*Nx-2);
yy = ifft(yy, 2*Nx-2);
uu = xx(1:Nx);
uv = yy(1:Nx);
for q=1:Nx
    %su(1,q) = uu(1,q)
    %sv(1,q) = dt*sin(uv(1,q))+uu(1,q)
    %uu(1,q) = uu(1,q)
    uv(1,q) = -dt*sin(uu(1,q)) + uv(1,q) ;

    %For the Plot
    FF(1,q)=abs(uu(1,q));
    FF(2,q)=abs(uv(1,q));
    %%%%%%%%%%%%%%%
end
xx = fft([uu uu(end-1:-1:2)],2*Nx-2);
yy = fft([uv uv(end-1:-1:2)],2*Nx-2);
qq = fftshift(xx);
qz = fftshift(yy);

%5. second t/2 part

for p=1:2*Nx-2
    uuu(1,1) = qq(1,p);
    uuu(1,2) = qz(1,p);

    k = (pi/L)*(-Nx+p);
    Mmat = [0 1; -k^2 0];

    uuu = expm(Mmat*dt/2)*transpose(uuu);

    qq(1,p) = uuu(1,1);
    qz(1,p) = uuu(2,1);
end

%print out curr. time
r

```

```

if(mod(r,5)==0)

    % backtransforming into real space
    xx = ifftshift(qq);
    yy = ifftshift(qz);
    xx = ifft(xx, 2*Nx-2);
    yy = ifft(yy, 2*Nx-2);
    uu = xx(1:Nx);
    uv = yy(1:Nx);

    figure(1)
    axis([xmin xmax ymin ymax])
    plot(x,uu)
    hold on
    axis([xmin xmax ymin ymax])
    plot(x,uv, 'r')

    %plot exact sol.
    % t=(r+2)*dt;
    % F(1,:) = 4*atan(const*exp((x-L/2-lambda*t-lambda*versetz)/qu));
    % F(2,:) = -4*lambda*const*exp((x-L/2-lambda*t-lambda*versetz)/qu)./(qu*(1+const^2*exp(2*(x-L/2-1
    % axis([xmin xmax ymin ymax])
    % plot(x,F(1,:), '--')
    % axis([xmin xmax ymin ymax])
    % plot(x,F(2,:), 'r--')
    % axis([xmin xmax ymin ymax])
    hold off
    title(sprintf('t=%g',t))
    pause(0.01)
    M(f)=getframe;
    f=f+1;
end

end

%showing & saving the movie
movie(M ,1)
movie2avi(M, 'kinkantikinkpairgeneration.avi')

```

## References

- [1] P.G. Drazin, R.S. Johnson, *Solitons: an introduction*, Cambridge University Press, ISBN 0-521-33655-4, 1990
  
- [2] M. Peirard, T. Dauxois, *Physique des solitons* , EDP Sciences-CNRS Editions, ISBN 2-271-06267-5, 2004
  
- [3] H. Ibach, H. Lueth, *Festkoerperphysik* , Springer-Verlag, ISBN 2-271-06267-5, 2002