General Introduction (.pdf slides)

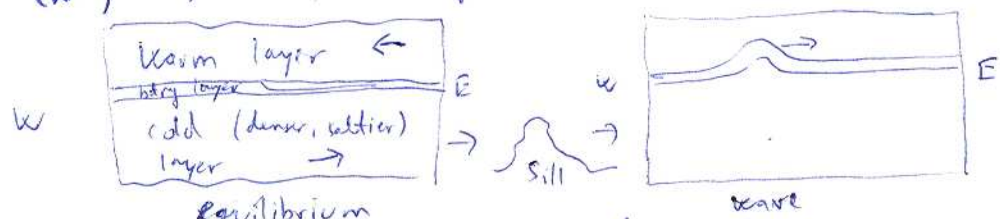
- movies of kdV 1-soliton and 2-soliton

kdV : $u_t + u u_x + u_{xxx} = 0$

- Russell's wave of translation
  - quotation (wikipedia) - 1834
  - Russell channel in Edinburgh

- tidal bores (kdV) (Severn in UK)
  - necessary conditions
    - large tidal range
    - tide funnelled into a shallow, narrowing river from a broad bay

- internal waves (kdV, mkdV) (Strait of Gibraltar - Camarinal Sill)



- depth ~ 100 m
- wave amplitude up to 100 m

water waves = extreme internal waves (water/air interface)

- morning glory (kdV) - Gulf of Carpentaria (N Australia)
  - sea breeze from East and another from W collide over highlands of Cape York Peninsula => bore generated - waves ordered by amplitude
  - roll clouds accompany these internal waves only sometimes

- freak waves (NLS)
  - not very well understood
  - associated to modulational instability

Solitary waves :
- permanent form
- localised - approach zero or another constant at infinity

Solitons :
- solit. waves with elastic collisions - except for phase & position shift

- no "iff condition" for soliton existence
- solitary waves - balance of nonlinearity (steepening (fluids), focusing (electromagnetism)) and dispersion
- often PDEs with solitons are completely integrable

counter example: Benjamin - Bona - Mahoney eqn. $u_t + u u_x - u_{xxt} = 0$ has soliton and is not integrable by IST

- no "iff cond." of integrability (unlike Hamiltonian ODEs)
- necess. cond. : inf. nr. of conserved quantities
- for IST-integrable systems u is a soliton iff it generates only points' spectrum in the direct scattering

Soliton eqn examples:

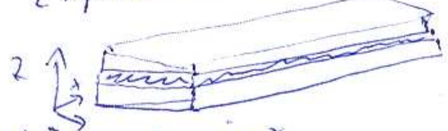
1) KdV (shallow water, internal waves, ...)

$$u_t + u u_x + u_{xxx} = 0$$

2) SG (Josephson junction, SIT, ...)

$$u_{tt} - u_{xx} + \sin u = 0$$

Joseph junc. : 2 superconductors linked by an insulator



TEM field propagates in x

SIT - Short intense pulses of light propagating through a 2-level medium which is in ground state before incidence - atoms excited by pulse front and release the energy back to the tail of the pulse

3) NLS (pulses in optical fibers, deep water waves, ...)

$$i u_t + u_{xx} + |u|^2 u = 0$$

Solitary wave eqn examples:

1) CQNLS (plasma waves, fiber lasers)

$$i u_t + u_{xx} + |u|^2 u - \mu |u|^4 u = 0$$

2) NLS with saturable nonlinearity (~~more accurate model~~ fibers with saturable nonlinearity)

$i u_t$

3) gKdV ($K(m,n)$ eqn.) - compactons

$$u_t + (u^m)_x + (u^n)_{xxx} = 0$$

4) CME - ~~light pulses in~~ - wavepackets of light in fiber Bragg gratings

$$i(u_t + u_z) + \kappa v + (|u|^2 + 2|v|^2) u = 0$$
$$i(v_t - v_z) + \kappa u + (|v|^2 + 2|u|^2) v = 0$$

Simple wave models (linear)

1) advection equ. $\quad u_t + c\,u_x = 0$, $t \geq 0,\ x \in \mathbb{R}$

$$u(x,0) = f(x)$$
$$\Rightarrow u(x,t) = f(x - ct)$$

2) wave equation $\quad u_{tt} - c^2 u_{xx} = 0$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$
$$\Rightarrow u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

3) shallow water small amplitude waves:

$$u_t + c\,u_x + u_{xxx} = 0$$
$$u(x,0) = f(x) \in L^2(\mathbb{R})$$

solution: $\hat{u}(k,t) = \frac{1}{2\pi}\int_{\mathbb{R}} u(x,t)\, e^{-ikx}\, dx$

$$\hat{u}_t + c\,ik\,\hat{u} - ik^3\hat{u} = 0 \quad , \quad \hat{u}(k,0) = \hat{f}(k)$$
$$\Rightarrow \hat{u}(k,t) = e^{-i(ck-k^3)t}\,\hat{f}(k) =: e^{-i\omega(k)t}\,\hat{f}(k)$$
$$u(x,t) = \int e^{i(kx - \omega(k)t)}\,\hat{f}(k)\,dk \quad , \quad \omega(k) = ck - k^3 \text{ disp. rel.}$$

In general, for $P(\partial_t, \partial_x)u = 0$, P polynomial (degree 1 in ∂_t) to obtain the disp. rel. it is enough to substitute $e^{i(kx-\omega t)}$ $\quad u = e^{ikx-\omega t}$

$\rightarrow$ obtain $\quad P(-i\omega, ik) = 0$

whose solution is denoted by $\omega = \omega(k)$

For general const. coeff. linear PDE 1^{st} order in t: $\quad$ (if higher order in t, rewrite as a larger system)

$$P(\partial_t, \partial_{x_1}\ldots\partial_{x_m})\,\vec{u} = 0 \quad , \quad f, u \in \mathbb{R}^n,\ x \in \mathbb{R}^m \quad , P \ldots \text{deg. 1 in } \partial_t$$
$$\vec{u}(\vec{x},0) = \vec{f} \qquad P: \mathbb{R}^{m+1} \to \mathbb{R}^{n\times n} \text{ (}n\times n\text{ matrix)}$$

$$\Rightarrow P(\partial_t, i\vec{k})\,\hat{\vec{u}}(\vec{k},t) = 0$$

disp. relation: $\quad \det P(-i\omega, i\vec{k}) = 0 \quad \ldots \quad n$ roots $\omega = \omega_{1\ldots n}(\vec{k})$ (~~ev~~ evals of $-iP(0, ik)$)

$$\vec{u}(x,t) = \int_{\mathbb{R}^m} e^{i\vec{k}\cdot\vec{x}}\, V \begin{pmatrix} e^{-i\omega_1 t} & & \\ & \ddots & \\ & & e^{-i\omega_n t}\end{pmatrix} V^{-1}\,\hat{\vec{f}}(\vec{k})\,d\vec{k}$$

WLOG. Consider $n = 1$

i.e. $\quad u(\vec{x},t) = \int e^{i(\vec{k}\cdot\vec{x} - \omega(\vec{k})t)}\,\hat{f}(\vec{k})\,d\vec{k} \qquad (1)$

and only ω real (i.e. waves)

2 notions of velocity:

phase velocity v_p: $\frac{\omega}{k}$ for $m=1$

$\frac{\omega}{|\vec{k}|^2}\vec{k}$ for $m>1$

group velocity v_g: $\nabla\omega(\vec{k})$

$f \in L^2 \Rightarrow$ (1) is a wave train

Define phase: $\theta = \vec{k}\cdot\vec{x} - \omega t$

$$u = \int_{\mathbb{R}^m} e^{i\theta(\vec{x},t)} \hat{f}(\vec{k})\, d\vec{k}$$

- phase surfaces $\theta = \text{const.}$ move at v_p
- as shown below: wavenumbers propagate at v_g

roughly ~~speaking~~: different wave numbers propagate at different velocities $\Rightarrow$ system is dispersive

Df: $\det\left|\frac{\partial\omega}{\partial k_i \partial k_j}\right| \neq 0 \iff$ system dispersive.

Note: dispersion causes any initially localized hump to disintegrate
$\to$ need for nonlinearity to obtain solitary evolution.

Note: why (for $m=1$) $\omega''(k)$ is stronger than $v_p' \neq 0$

ex: $\omega = ak+b$
$\Rightarrow v_p' = a \neq 0$
$\omega''(k) = 0$

$u = \int e^{-ibt} e^{ik(x-at)} \hat{f}(k)$
$= e^{-ibt} f(x-at)$
— no dispersion

Phase velocity vs. group velocity $(m=1)$

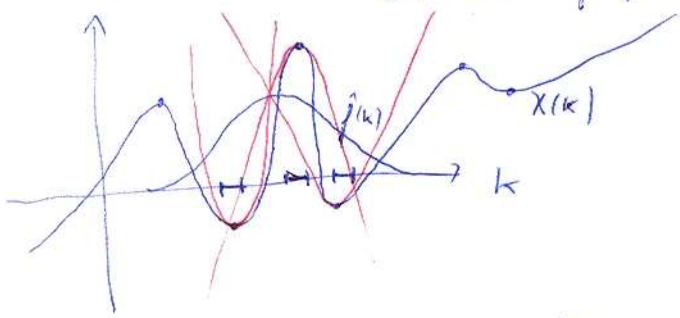
- consider the limit $t \to \infty$ with $\frac{x}{t} = \text{const.} = c$.

(waves along the ray $x = ct$)

Let $u = \int \hat{f}(k)\, e^{-i\chi(k)t}\, dk$, $\chi = \omega(k) - k \underbrace{\frac{x}{t}}_{=c} = -\frac{\theta}{t}$

as $t \to \infty$ $e^{-i\chi(k)t}$ is highly oscillatory (in k)

method of stationary phase : main contribution to u is from extrema of χ : ~~[illegible]~~



$\chi'(\tilde{k}) = 0 \iff \omega'(\tilde{k}) = \frac{x}{t}$

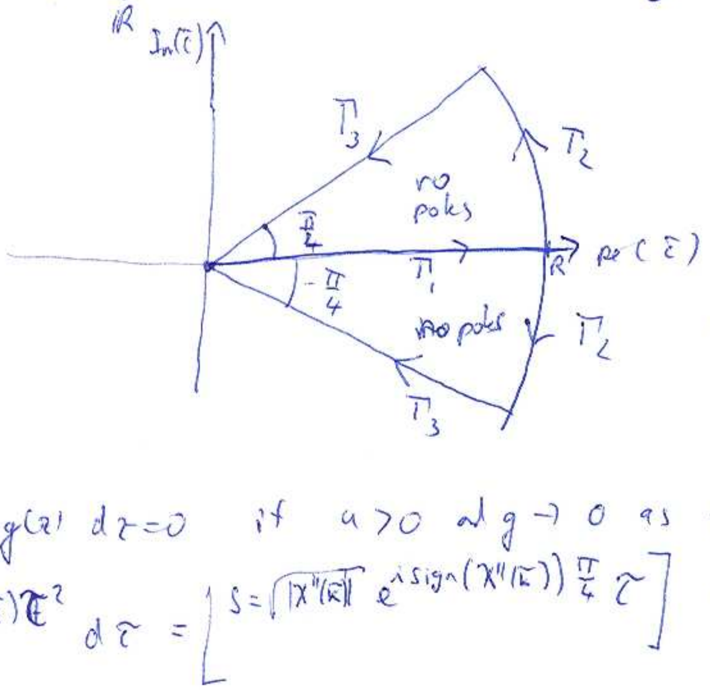
$$u \approx \sum_{\chi'(\tilde{k})=0} \hat{f}(\tilde{k})\, e^{-i\chi(\tilde{k})t} \underbrace{\int_{\mathbb{R}} e^{-\frac{i}{2}(k-\tilde{k})^2 \chi''(\tilde{k})t}\, d\tilde{k}}_{=:\, I} \quad , \text{if } \chi''(\tilde{k}) \neq 0$$

we used $\hat{f}(k) \sim \hat{f}(\tilde{k})$ b/c $\hat{f}$ varies much slower than $e^{i\chi(k)t}$

Calculate I $\left(\text{use } \int_{\mathbb{R}} e^{-\alpha y^2}\, dy \overset{(*)}{=} \left(\frac{\pi}{\alpha}\right)^{1/2} \right)$

$$I = \left[\begin{array}{l} \tau := \sqrt{\frac{t}{2}}\,(k-\tilde{k}) \\ d\tau = \sqrt{\frac{t}{2}}\, dk \end{array} \right] = \sqrt{\frac{2}{t}} \int_{\mathbb{R}} e^{-i\chi''(\tilde{k})\tau^2}\, d\tau = 2\left(\frac{2}{t}\right)^{\frac{1}{2}} \int_0^{\mathbb{R}} e^{-i\chi''(\tilde{k})\tau^2}\, d\tau$$

contour integration : to reduce the problem to $(*)$



$\leftarrow$ for $\chi''(\tilde{k}) < 0$

$\leftarrow$ for $\chi''(\tilde{k}) > 0$

$$I = -\frac{2\sqrt{2}}{\sqrt{t}} \lim_{R\to\infty} \left(\int_{\Gamma_2} + \int_{\Gamma_3} \right)$$

Jordan's lemma: $\lim_{R\to\infty} \int_{\Gamma_2} e^{iaz} g(z)\, dz = 0$ if $a > 0$ and $g \to 0$ as $z \to \infty$.

$$\Rightarrow \quad I = -2\left(\frac{2}{t}\right)^{\frac{1}{2}} \int_{\Gamma_3} e^{-i\chi''(\tilde{k})\tau^2}\, d\tau = \left[s = \sqrt{|\chi''(\tilde{k})|}\, e^{i\,\mathrm{sign}(\chi''(\tilde{k}))\frac{\pi}{4}}\, \tau \right] =$$

$$= -2\left(\frac{2}{t\,|\chi''(\tilde{k})|}\right)^{1/2} e^{-i\,\mathrm{sign}(\chi''(\tilde{k}))\frac{\pi}{4}} \int_\infty^0 e^{-s^2}\,ds$$

$$= \left(\frac{2\pi}{t\,|\chi''(\tilde{k})|}\right)^{1/2} e^{-i\,\mathrm{sign}(\chi''(\tilde{k}))\frac{\pi}{4}}$$

note: $\chi'' = \omega''$, $\chi(k)t = \omega(k)t - kx$

$$\therefore \quad u \sim \sum_{\omega'(\tilde{k})=c} \left(\frac{2\pi}{t\,|\omega''(\tilde{k})|}\right)^{1/2} \hat{f}(\tilde{k})\, e^{i(\tilde{k}x-\omega(\tilde{k})t)}\, e^{-i\,\mathrm{sign}(\chi''(\tilde{k}))\frac{\pi}{4}}$$

i.e. the main contribution along $\frac{x}{t}=c$ comes from wavenumbers $\tilde{k}$ with $\omega'(\tilde{k})=c$!

→ Observer moving at vel. $\omega'(k_0)$ sees waves with wavenr. k_0 and freq. $\omega(k_0)$ but ~~keep~~ crests keep passing him

→ . . . . . at vel. $v_p = \frac{\omega}{k}$ sees the same crest but local wavenr. & freq. keep changing, i.e. neighboring crests get further or closer.

<u>Local wavenr. & local freq.</u>:

$\tilde{k}$ ~~is a~~ fn of (x,t) – given by $\omega'(\tilde{k}) = \frac{x}{t}$

disp. rel then gives ω as a fn of (x,t):

$$\tilde{\omega} = \omega(\tilde{k}) = \omega(\tilde{k}(x,t))$$

We call $\tilde{k}, \tilde{\omega}$ ... local wavenr. & freq.

$\theta(x,t) := x\,\tilde{k}(x,t) - t\,\omega(\tilde{k})$

so $u \sim \sum_{\tilde{k}} A_{\tilde{k}}\, \underbrace{e^{i\theta(x,t)}}_{\text{nonuniform}}$ plane wave

note: $\theta_x = \tilde{k} + \underbrace{(x-\omega'(\tilde{k})t)}_{=0}\tilde{k}_x = \tilde{k}$

$\theta_t = -\omega(\tilde{k}) + \underbrace{(x-\omega'(\tilde{k})t)}_{=0}\tilde{k}_t = -\tilde{\omega}$

So local wavenr. & freq. agree with standard wave nr. & freq. (are consistent)

Smoothing through dispersion

easy special case: $i u_t = u_{xx}$ ... linear Schröd. equ.

heat equ: $u_{\tilde t} = u_{xx}$

$$u(x,\tilde t) = \frac{1}{\sqrt{4\pi \tilde t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\tilde t}} u(y,0)\,dy$$

diffusion of energy: $\partial_t \int_{\mathbb{R}} u^2\,dx = 2\int u u_t\,dx = 2\int u u_{xx}\,dx = -2\int u_x^2\,dx < 0$

lin. Schrödinger: let $t = i\tilde t$ $\quad (\partial_{\tilde t} = i\partial_t)$

$$u(x,t) = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} e^{-\frac{i(x-y)^2}{4t}} u(y,0)\,dy$$

energy conserved: $\partial_t \int_{\mathbb{R}} |u|^2 dx = \int u^* u_t + u u_t^* = \int -i u^* u_{xx} + i u u^*_{xx}$

$= i\int |u_x|^2 - |u_x|^2 = 0$

smoothing: $$\partial_x u(x,t) = \frac{1}{\sqrt{\pi i t}} \int_{\mathbb{R}} -\frac{2i(x-y)}{4t}\, e^{\frac{i(x-y)^2}{4t}} u(y,0)\,dy$$

... exists $\forall x \in \mathbb{R}, t > 0$ if $\int |x||u(x,0)|\,dx < \infty$

i.e. existence of $\partial_x u(x,0)$ not necessary!

explanation: nonsmooth parts consist of high Four. modes as well as low. the higher modes propagate faster to inf. so the singularity disperses

(Non)Dispersive systems:

- advection equ. $\quad u_t + c u_x = 0$
  $-i\omega + c i k = 0$
  $\omega = ck \quad \Rightarrow \omega'' = 0$
- wave equ: $\quad u_{tt} - c^2 u_{xx} = 0$
  $-\omega^2 + c^2 k^2 = 0$
  $\omega_1 = ck$
  $\omega_2 = -ck \quad \Rightarrow \omega'' = 0$
- linear KdV $\quad u_t + c u_x + u_{xxx} = 0$
  ~~xxxx~~ $\omega = ck - k^2 \quad \Rightarrow \omega'' = -2 \neq 0$
- beam equ.

Multiple scales expansions

- for asymptotic expansions of solutions with more scales of variation (in t or x)

ex: $u_{tt} + 2\varepsilon u_t + u = 0$ ... lin. oscillator w/ small damping

$u(0) = a$ $\quad 0 < \varepsilon \ll 1$

$u_t(0) = 0$

exact sol.: $u = A e^{-\varepsilon t} \cos\left(\sqrt{1-\varepsilon^2}\, t + \phi\right)$

$A = \frac{a}{\sqrt{1-\varepsilon^2}}$, $\phi = -\arctan\left(\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right)$

try to solve via regular perturb. exp.:

$u \approx u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t)$

$O(1)$ $\left.\begin{array}{l} u_{0tt} + u_0 = 0 \\ u_0(0) = a \\ u_{0t}(0) = 0 \end{array}\right\} \Rightarrow \underline{u_0(t) = a\cos(t)}$

$O(\varepsilon)$: $\left.\begin{array}{l} u_{1tt} + u_1 = -2u_{0t} = 2a\sin t \\ u_1(0) = u_{1t}(0) = 0 \end{array}\right\} \Rightarrow \underline{u_1(t) = -at\cos t + a\sin}$

linear growth in t !

So $\left|\frac{u_1(t)}{u_0(t)}\right| = O(1)$ holds only for $t = O(1)$

$(\exists C \mid |\frac{u_1}{u_0}| < C \text{ as } \varepsilon \to 0)$ (not even $O(\frac{1}{\varepsilon})$)

$\Rightarrow$ expansion fails to be valid after $t = O(1)$

remedy: multiple scales expansion:

introduce $t_0 = t$ $\quad u(t) \sim u_0(t_0, t_1, \dots) + \varepsilon u_1(t_0, t_1, \dots) +$

$t_1 = \varepsilon t$

$t_2 = \varepsilon^2 t$

aim: determine u_0, u_1 and their dependence on t_0, t_1

$\frac{d}{dt} = \partial_{t_0} + \varepsilon \partial_{t_1} + \dots$, $\frac{d^2}{dt^2} = \partial_{t_0}^2 + 2\varepsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2 \partial_{t_1}^2 + \dots$

$O(1)$: $\partial_{t_0}^2 u_0 + u_0 = 0$, $\left.\begin{array}{l} u_0(0,0) = a \\ u_{0t_0}(0,0) = 0 \end{array}\right\} \Rightarrow$

$\Downarrow$ $\qquad\qquad \Downarrow$ $A(0)\cos(\phi(0)) = a$

$\underline{u_0 = A(t_1)\cos(t_0 + \phi(t_1))}$ $\qquad -A(0)\sin(\phi(0)) = 0$

so $\phi(0) = 0$

$\underline{A(0) = a}$