The Kortweg-de Vries Equation

$$u_t + uu_x + 6u_{xxx} = 0$$

Claude Gittelson

12.12.2006

Outline

Contents

1 FPU Problem **1**
1.1 The Fermi-Pasta-Ulam Problem 1
1.2 Continuum Approximation of FPU 2

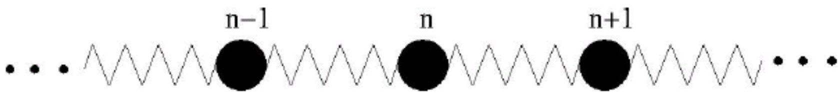
2 Shallow Water Waves **3**
2.1 Problem Setup 3
2.2 Governing Equations 3
2.3 Multiple Scales Expansion 4

3 Exact Solutions of KdV **6**
3.1 Phase Plane Analysis 6
3.2 Cnoidal Waves 6

4 Numerical Computation of KdV 1-Soliton **7**
4.1 Petviashvili Iteration 7

1 Continuum Approximation of the Fermi-Pasta-Ulam Problem

1.1 The Fermi-Pasta-Ulam Problem



Consider a line of particles of mass m at positions $(y_n)_{n\in\mathbb{Z}}$ connected by springs with *nonlinear* coupling forces

$$F_{n,n+1} = k(y_{n+1} - y_n) + \alpha k(y_{n+1} - y_n)^2.$$

The corresponding equations of motion are

$$m\partial_t^2 y_n = k\left((y_{n+1} - 2y_n + y_{n-1}) + \alpha(y_{n+1} - y_n)^2 + \alpha(y_n - y_{n+1})^2\right).$$

1.2 Continuum Approximation of FPU

Define equilibrium positions $x_n = nh$, lattice spacing h, displacement $y(x)$ with $y(x_n) = y_n$. A Taylor series gives

$$\begin{aligned} y_{n+1} &= y_n + hy_x(x_n) + \frac{h^2}{2}y_{xx}(x_n) + \frac{h^3}{6}y_{xxx}(x_n) + \frac{h^4}{24}y_{xxxx}(x_n) + \mathcal{O}\left(h^5\right) \\ y_{n-1} &= y_n - hy_x(x_n) + \frac{h^2}{2}y_{xx}(x_n) - \frac{h^3}{6}y_{xxx}(x_n) + \frac{h^4}{24}y_{xxxx}(x_n) + \mathcal{O}\left(h^5\right). \end{aligned}$$

Therefore,

$$y_{n+1} - 2y_n + y_{n-1} = h^2 y_{xx}(x_n) + \frac{h^4}{12}y_{xxxx}(x_n) + \mathcal{O}\left(h^5\right)$$

and

$$\begin{aligned} (y_{n+1} - y_n)^2 - (y_n - y_{n-1})^2 =& (y_{n+1} - y_{n-1})(y_{n+1} - 2y_n + y_{n-1}) \\ =& (2hy_x(x_n) + \mathcal{O}\left(h^3\right))(h^2 y_{xx}(x_n) + \mathcal{O}\left(h^4\right)) \\ =& 2h^3 y_x(x_n) y_{xx}(x_n) + \mathcal{O}\left(h^5\right). \end{aligned}$$

The equations of motion become the pde

$$y_{tt} = \frac{kh^2}{m}\left(y_{xx} + 2\alpha h y_x y_{xx} + \frac{1}{12}h^2 y_{xxxx}\right) + \mathcal{O}\left(h^5\right).$$

Assume $\alpha \sim h$ small. The dominant part of the equation is

$$y_{tt} = \frac{kh^2}{m} y_{xx}.$$

d'Alembert $\Longrightarrow y = f(x - vt) + g(x + vt)$, $v^2 = \frac{kh^2}{m}$.

Let's study the right-going wave f. Define $X = x - vt$ and slow time $T = \alpha hvt$. Then

$$\partial_x = \frac{\partial X}{\partial x}\partial_X + \frac{\partial T}{\partial x}\partial_T = \partial_X, \quad \partial_t = \frac{\partial X}{\partial t}\partial_X + \frac{\partial T}{\partial t}\partial_T = -v\partial_X + \alpha hv\partial_T$$

Therefore

$$\partial_t^2 = v^2\partial_X^2 - 2\alpha hv^2\partial_X\partial_T + \alpha^2 h^2 v^2 \partial_T^2.$$

The equation of motion becomes

$$y_{XX} - 2\alpha h y_{XT} + \alpha^2 h^2 y_{TT} = y_{XX} + 2\alpha h y_X y_{XX} + \frac{1}{12}h^2 y_{XXXX}.$$

Since the y_{TT}-term is of *higher order* than the others, we have

$$y_{XT} + y_X y_{XX} + \frac{h}{24\alpha} y_{XXXX} = 0.$$

Introducing $u = y_X$ and $\delta^2 = \frac{h}{24\alpha}$ leads to the *Kortweg-de Vries* (KdV) equation

$$u_T + uu_X + \delta^2 u_{XXX} = 0.$$

2 Shallow Water Waves

2.1 Problem Setup

h depth of water in equilibrium

$\tilde{\zeta}(\tilde{x}, \tilde{t})$ surface of water, i.e. depth of waver is $\tilde{\zeta} + h$

$\tilde{\phi}(\tilde{x}, \tilde{z}, \tilde{t})$ velocity potential, i.e. velocity of water is $(\tilde{\phi}_{\tilde{x}}, \tilde{\phi}_{\tilde{z}})$

2.2 Governing Equations

$\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{z}\tilde{z}} = 0$ for $-h < \tilde{z} < \tilde{\zeta}(\tilde{x}, \tilde{t})$ (incompressible potential flow)

$\tilde{\phi}_{\tilde{z}} = 0$ at $\tilde{z} = -h$ (no flow into bottom)

$\tilde{\zeta}_{\tilde{t}} + \tilde{\phi}_{\tilde{x}} \tilde{\zeta}_{\tilde{x}} = \tilde{\phi}_{\tilde{z}}$ at $\tilde{z} = \tilde{\zeta}(\tilde{x}, \tilde{t})$ (kinematic equation)

$\tilde{\phi}_{\tilde{t}} + g\tilde{\zeta} + \frac{1}{2}(\tilde{\phi}^2_{\tilde{x}} + \tilde{\phi}^2_{\tilde{z}}) = 0$ at $\tilde{z} = \tilde{\zeta}(\tilde{x}, \tilde{t})$ (dynamic condition)

Linearizing around the equilibrium solutions $\tilde{\zeta}(\tilde{x}, \tilde{t}) = 0$ and $\tilde{\phi}(\tilde{x}, \tilde{z}, \tilde{t}) = 0$ leads to the linear dispersion relation

$$\omega^2 = gk \tanh(kh).$$

Take $\tilde{\zeta}(\tilde{x}, \tilde{t}) = \zeta_0 \, \mathrm{e}^{\mathrm{i}(\omega\tilde{t} - k\tilde{x})}$ and $\tilde{\phi}(\tilde{x}, \tilde{z}, \tilde{t}) = \phi_0(\tilde{z}) \, \mathrm{e}^{\mathrm{i}(\omega\tilde{t} - k\tilde{x})}$. In the linearized equations, the nonlinear terms are simply dropped. The first two equations imply $\phi_0(\tilde{z}) = a\cosh(k(\tilde{z} + h))$. By the third equation, $\zeta_0 = -ia\frac{k}{\omega}\sinh(kh)$ and the dispersion relation follows from the last equation.

Consider long waves with small amplitude, ie.

$$(kh)^2 \approx \epsilon \approx h^{-1} \max|\zeta|$$

where k is the wave number in $\tilde{x}$-direction. The linear dispersion relation $\omega^2 = gk\tanh(kh)$ becomes

$$\omega^2 = gk^2 h.$$

Switch to dimensionless variables

$$\begin{aligned} \tilde{\zeta} &= \epsilon h \zeta \sim \max|\tilde{\zeta}|\zeta \\ \tilde{\phi} &= h\sqrt{\epsilon g h}\phi \sim h^2 \omega \phi \end{aligned}$$

$$\begin{aligned} x &= \sqrt{\epsilon}\frac{\tilde{x}}{h} \sim k\tilde{x} \\ t &= \sqrt{\frac{\epsilon g}{h}}\tilde{t} \sim \omega\tilde{t} \end{aligned}$$

$$z = \frac{\tilde{z}}{h}$$

The equations become

$$\begin{cases} \phi_{zz} + \epsilon\phi_{xx} = 0, & \text{for} \quad -1 < z < \epsilon\zeta(x,t) \\ \phi_z = 0, & \text{at} \quad z = -1 \\ \epsilon\zeta_t + \epsilon^2\phi_x\zeta_x = \phi_z, & \text{at} \quad z = \epsilon\zeta(x,t) \\ \phi_t + \zeta + \dfrac{1}{2}\phi_z^2 + \dfrac{\epsilon}{2}\phi_x^2 = 0, & \text{at} \quad z = \epsilon\zeta(x,t) \end{cases}$$

2.3 Multiple Scales Expansion

Expand ϕ to

$$\phi(x,z,t) = \sum_{n=0}^{\infty}(z+1)^n \phi^n(x,t).$$

By the first equation,

$$\epsilon \sum_{n=0}^{\infty}(z+1)^n \phi^n_{xx}(x,t) + \sum_{n=2}^{\infty}(z+1)^{n-2} n(n-1)\phi^n(x,t) = 0,$$

so

$$\phi^n(x,t) = \frac{-\epsilon \phi^{n-2}_{xx}(x,t)}{n(n-1)} \quad \forall n.$$

Since $\phi_z(x,-1,t) = \phi^1(x,t)$, the second equation implies

$$0 = \phi^1 = \phi^3 = \phi^5 = \dots$$

Therefore,

$$\phi(x,z,t) = \phi^0(x,t) - \epsilon \frac{(z+1)^2}{2}\phi^0_{xx}(x,t) + \epsilon^2 \frac{(z+1)^4}{4!}\phi^0_{xxxx}(x,t) + \dots$$

and

$$\phi_z(x, \epsilon\zeta(x,t), t) = -\epsilon(1+\epsilon\zeta(x,t))\phi^0_{xx}(x,t) + \epsilon^2 \frac{(1+\epsilon\zeta(x,t))^3}{6}\phi^0_{xxxx}(x,t) + \dots.$$

Expand ζ, ϕ^0_x and t

$$\begin{aligned} \zeta &= \zeta^0 + \epsilon \zeta^1 \\ \phi^0_x &= u^0 + \epsilon u^1 \\ \tau_0 &= t, \quad \tau_1 = \epsilon t \qquad \Longrightarrow \qquad \partial_t = \frac{\partial \tau_0}{\partial t}\partial_{\tau_0} + \frac{\partial \tau_1}{\partial t}\partial_{\tau_1} = \partial_{\tau_0} + \epsilon \partial_{\tau_1} \end{aligned}$$

It follows that

$$\begin{aligned} \phi_z(x,\epsilon\zeta(x,t),t) &= -\epsilon u^0_x + \epsilon^2 \left(-\zeta^0 u^0_x - u^1_x + \frac{1}{6}u^0_{xxx}\right) + \mathcal{O}\left(\epsilon^3\right) \\ \phi_x(x,\epsilon\zeta(x,t),t) &= u^0 + \epsilon \left(u^1 - \frac{1}{2}u^0_{xx}\right) + \mathcal{O}\left(\epsilon^2\right) \\ \phi_{xt}(x,\epsilon\zeta(x,t),t) &= u^0_{\tau_0} + \epsilon \left(u^0_{\tau_1} + u^1_{\tau_0} - \frac{1}{2}u^0_{xx\tau_0}\right) + \mathcal{O}\left(\epsilon^2\right) \\ \zeta_t(x,t) &= \zeta^0_{\tau_0} + \epsilon\left(\zeta^1_{\tau_0} + \zeta^0_{\tau_1}\right) + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

Kinematic equation $\epsilon\zeta_t + \epsilon^2 \phi_x \zeta_x = \phi_z, \quad z = \epsilon\zeta(x,t)$

$\mathcal{O}\left(\epsilon\right)$ $\quad \zeta^0_{\tau_0} + u^0_x = 0$

$\mathcal{O}\left(\epsilon^2\right)$ $\quad \zeta^1_{\tau_0} + u^1_x = -\left(\zeta^0_{\tau_1} + u^0\zeta^0_x + \zeta^0 u^0_x - \frac{1}{6}u^0_{xxx}\right)$

Bernoulli's equation $\phi_{xt} + \zeta_x + \frac{1}{2}\partial_x \left((\phi_z)^2 + \epsilon(\phi_x)^2\right) = 0, \quad z = \epsilon\zeta(x,t)$

$\mathcal{O}(1) \quad u^0_{\tau_0} + \zeta^0_x = 0$

$\mathcal{O}(\epsilon) \quad u^1_{\tau_0} + \zeta^1_x = -\left(u^0_{\tau_1} - \frac{1}{2}u^0_{xx\tau_0} + u^0 u^0_x\right)$

The lower order equations are the linear wave equation

$$\begin{cases} \zeta^0_{\tau_0} + u^0_x = 0 \\ u^0_{\tau_0} + \zeta^0_x = 0 \end{cases}$$

By d'Alembert, the solution is

$$\begin{cases} \zeta^0 = f(x-\tau_0,\tau_1) + g(x+\tau_0,\tau_1) \\ u^0 = f(x-\tau_0,\tau_1) - g(x+\tau_0,\tau_1) \end{cases}$$

Introduce characteristic variables $l = x + \tau_0$ and $r = x - \tau_0$.

$$\partial_x = \partial_l + \partial_r \quad \text{and} \quad \partial_{\tau_0} = \partial_l - \partial_r$$

$$\zeta^0 = f(r) + g(l)\ , \qquad u^0 = f(r) - g(l)$$

The higher order equations become

$$\begin{cases} \zeta^1_l - \zeta^1_r + u^1_l + u^1_r = \\ \quad -\left(f_{\tau_1} + g_{\tau_1} + (f-g)(f_r+g_l) + (f+g)(f_r-g_l) - \frac{1}{6}(f_{rrr} - g_{lll})\right) \\ u^1_l - u^1_r + \zeta^1_l + \zeta^1_r = -\left(f_{\tau_1} - g_{\tau_1} + \frac{1}{2}(f_{rrr} + g_{lll}) + (f-g)(f_r - g_l)\right) \end{cases}$$

Adding and subtracting these, and integrating once, leads to

$$\begin{cases} 2(\zeta^1 + u^1) = -\left(2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr}\right) l + f_r \int g dl + \frac{1}{2}g^2 + fg - \frac{2}{3}g_{ll} + c_1 \\ 2(u^1 - \zeta^1) = -\left(2g_{\tau_1} - 3gg_l - \frac{1}{3}g_{lll}\right) r - g_l \int f dr - \frac{1}{2}f^2 - gf + \frac{2}{3}f_{rr} + c_2 \end{cases}$$

Since all terms except the first are bounded for $l, r \to \pm\infty$,

$$\begin{cases} 2f_{\tau_1} + 3ff_r + \frac{1}{3}f_{rrr} = 0 \\ 2g_{\tau_1} - 3gg_l - \frac{1}{3}g_{lll} = 0 \end{cases} \qquad (KdV)$$

Similarities in Derivations of KdV

We have derived the KdV equation in the forms

$$\begin{aligned} u_T + uu_X + \delta^2 u_{XXX} &= 0 \\ 2f_{\tau_1} + 3ff_r + \tfrac{1}{3}f_{rrr} &= 0 \\ 2g_{\tau_1} - 3gg_l - \tfrac{1}{3}g_{lll} &= 0 \end{aligned}$$

Similarities include

- slow time (T, τ_1)
- moving frame of reference (X, r, l)
- unknown is a velocity ($u = y_X$, $\phi_x = f - g + \mathcal{O}(\epsilon)$)

The KdV equation defines the slowly varying shape of a wave.

3 Exact Solutions of KdV

3.1 Phase Plane Analysis

Consider stationary solutions $u(x,t) = U(x - vt)$ of the KdV equation

$$4u_t = 6uu_x + u_{xxx}$$

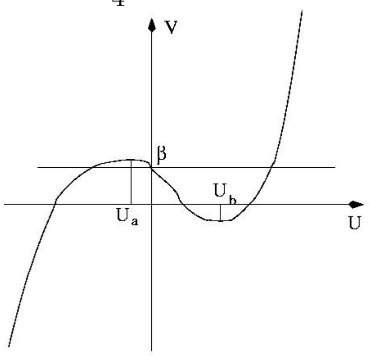
U satisfies $-4vU' = 6UU' + U'''$; integration leads to

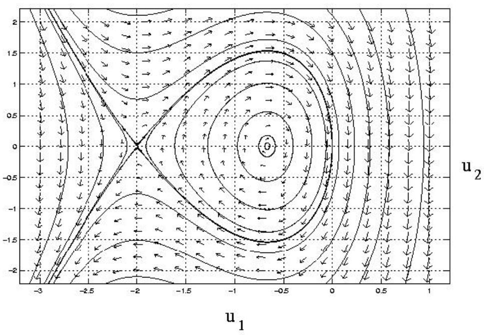
$$U'' = -(3U^2 + 4vU + \alpha) = -\frac{\partial V(U)}{\partial U}$$

the equation for a particle in a potential $V(U) = U^3 + 2vU^2 + \alpha U$.
First look for constant solutions ($U' = 0 = U''$), i.e. zeros of $\frac{\partial V}{\partial U}$.

$$\text{discriminant} \quad \Delta = 16v^2 - 12\alpha = 4(4v^2 - 3\alpha)$$

For $v^2 > \dfrac{3}{4}\alpha$, there are two constant solutions U_a and U_b.





The bounded solutions are

- periodic solutions around U_b with energy in $[V(U_b), V(U_a))$
- *solitons* on the separatrix at energy $V(U_a)$

Note: $U(\pm\infty) = 0 \;\Leftrightarrow\; U_a = 0, U_b > 0 \;\Leftrightarrow\; \alpha = 0, v < 0$ for soliton solutions.

3.2 Cnoidal Waves

We know that $U'' = -\frac{\partial V}{\partial U}$ for $V(U) = U^3 + 2vU^2 + \alpha U$. Multiplying by U' and integrating implies

$$\frac{1}{2}(U')^2 = E - V(U) \quad \text{or} \quad U' = \pm\sqrt{2(E - V(U))}$$

Therefore

$$\xi = \xi_0 \pm \int_{U(\xi_0)}^{U(\xi)} \frac{dU}{\sqrt{2(E - V(U))}}$$

Let $E - V(U) = (U_1 - U)(U_2 - U)(U_3 - U)$ with $U_1 \le U_2 \le U_3$ and assume $U(\xi_0) = U_3$. Then

$$\xi = \xi_0 \pm \int_{U_3}^{U(\xi)} \frac{dU}{\sqrt{2(U_1 - U)(U_2 - U)(U_3 - U)}}$$

Let $0 \leq m \leq 1$ and

$$v = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$$

Then

$$\mathrm{sn}(v|m) = \sin \phi \quad \text{and} \quad \mathrm{cn}(v|m) = \cos \phi$$

In particular, $\mathrm{cn}(v|0) = \cos v$ and

$$\mathrm{cn}(v|1) = \mathrm{sech}\, v = \frac{1}{\cosh v} = \frac{2}{\mathrm{e}^v + \mathrm{e}^{-v}}$$

Apply the substitution $U = U_3 - (U_3 - U_2)\sin^2 \theta$. Then

$$dU = -2(U_3 - U_2)\sin\theta\cos\theta$$

and

$$\begin{aligned}(U_1 - U)(U_2 - U)(U_3 - U) =& (U_1 - U_3 + (U_3 - U_2)\sin^2\theta)(U_2 - U_3 + (U_3 - U_2)\sin^2\theta)(U_3 - U_2)\sin^2\theta \\ =& (U_3 - U_1)(U_3 - U_2)^2 \sin^2\theta\cos^2\theta \left(1 - \frac{U_3 - U_2}{U_3 - U_1}\sin^2\theta\right)\end{aligned}$$

The implicit solution formula becomes

$$\xi = \xi_0 \mp \sqrt{\frac{2}{U_3 - U_1}} \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

with $m = \dfrac{U_3 - U_2}{U_3 - U_1}$ and

$$U(\xi) = U_3 - (U_3 - U_2)\sin^2\phi = U_2 + (U_3 - U_2)\cos^2\phi$$

By the definition of cn, $\mathrm{cn}\left((\xi - \xi_0)\sqrt{\frac{U_3 - U_1}{2}}\middle| m\right) = \cos\phi$, so

$$U(\xi) = U_2 + (U_3 - U_2)\,\mathrm{cn}^2\left((\xi - \xi_0)\sqrt{\frac{U_3 - U_1}{2}}\middle| m\right)$$

4 Numerical Computation of KdV 1-Soliton

4.1 Petviashvili Iteration

Stationary solution $u(x,t) = U(x - vt)$ with $U(\pm\infty) = 0$ ($\Leftrightarrow \ v < 0, \alpha = 0$) of KdV equation

$$4u_t - 6uu_x - u_{xxx} = 0$$

satisfies $4vU + U'' = -3U^2$. In frequency space,

$$(4v - 4\pi^2k^2)\widehat{U}(k) = -3\widehat{U^2}(k)$$

This suggests the iteration

$$\widehat{U}_{n+1}(k) = M_n^2 \frac{3\widehat{U_n^2}(k)}{(-4v) + 4\pi^2k^2}$$

with stabilizing factor

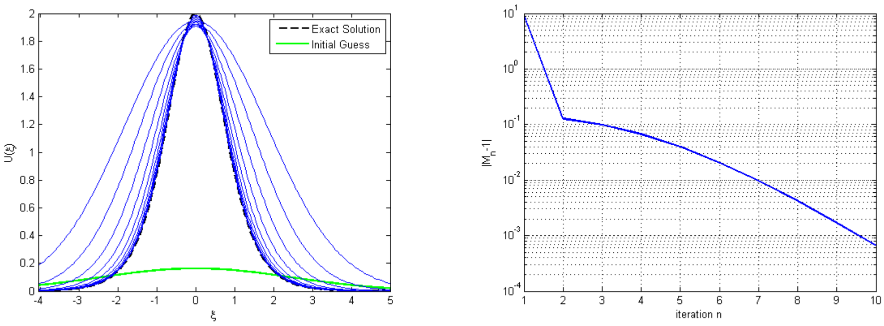
$$M_n = \frac{\frac{1}{3}\int((-4v) + 4\pi^2 k^2)\widehat{U}_n^2(k)dk}{\int \widehat{U}_n(k)\widehat{U_n^2}(k)dk}$$

The error can be estimated by $|M_n - 1|$.

KdV 1-soliton

$$U(\xi) = 2\,\mathrm{sech}(\xi)$$

for $v = -1$ computed via Petviashvili iteration from a Gaussian initial guess.



References

[1] B. Deconinck. *Solitons and Nonlinear Waves*, lecture notes, Sections 1.3, 3.1 and 4.2. University of Washington, 2006.

[2] P. G. Drazin and R. S. Johnson. *Solitons: an Introduction*, pages 22–30. Cambridge University Press, 1989.

[3] D. Pelinovsky and Y. Stepanyants. Convergence of Petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations. *SIAM J. Numer. Anal.*, 42(3):1110–1127, 2004.

[4] A. Scott. *Nonlinear Science*, pages 55–61. Oxford University Press, 2003.

[5] G. B. Whitham. *Linear and Nonlinear Waves*, pages 454–466. Wiley–Interscience, 1974.