# Theory and Numerics of Solitary Waves

## - Multiple Scales Method; NLS Derivation for Pulse Propagation in Optical Fibers -

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# Multiple Scales Expansion - motivation

multiple scales expansion = asymptotic expansion for problems with more scales in the dynamics (in time or space) [M.H.Holmes, Intro. do Perturb. Methods, Springer, 1995, Sec. 3]

**Example:** linear weakly damped oscillator

$u_{tt} + 2\varepsilon u_t + u = 0, \quad u(0) = a, u_t(0) = 0, \; 0 < \varepsilon << 1$

The exact solution: $u = \frac{ae^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}} \cos\left(\sqrt{1-\varepsilon^2}t - \arctan(\varepsilon/\sqrt{1-\varepsilon^2})\right).$

Suppose we do not know it and apply a regular perturbation expansion

$u \sim u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \dots$

We obtain at

- $\mathcal{O}(1)$ (as $\varepsilon \to 0$) $\qquad u_{0_{tt}} + u_0 = 0, \quad u_0(0) = a, u_{0_t}(0) = 0$

which has the solution $u_0 = a\cos(t)$

- $\mathcal{O}(\varepsilon) \qquad u_{1_{tt}} + u_1 = -2u_0(t) = 2a\sin(t), \quad u_1(0) = 0, u_{1_t}(0) = 0$

which has the solution $u_1 = -at\cos(t) + a\sin(t)$

Cleary, $u_1$ **grows linearly in time**. Thus the above asymptotic expansion fails at $t = 1/\varepsilon$ because that is when the condition $|u_{j+1}(t)/u_j(t)| = \mathcal{O}(1)$ ceases to hold.

# Multiple Scales Expansion

$$u_{tt} + 2\,\varepsilon\, u_t + u = 0, \quad u(0) = a, u_t(0) = 0,\ 0 < \varepsilon << 1$$

Introduce multiple time scales $t_0 = t, t_1 = \varepsilon\, t, t_2 = \varepsilon^2\, t, \ldots$ and propose the expansion:

$$u \sim u_0(t_0, t_1, \ldots) + \varepsilon\, u_1(t_0, t_1, \ldots) + \ldots$$

Our **aim**:determine $u_0$ and $u_1$ and only their dependence on $t_0$ and $t_1$.
Note (if $u_j = u_j(t_0, t_1)$):

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon\, \frac{\partial}{\partial t_1}, \qquad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_0^2} + 2\,\varepsilon\, \frac{\partial^2}{\partial t_1 \partial t_0} + \varepsilon^2\, \frac{\partial^2}{\partial t_1^2}$$

- $\mathcal{O}(1)$ $\quad \frac{\partial^2 u_0}{\partial t_0^2} + u_0 = 0, \quad u_0(0,0) = a, \frac{\partial u_0}{\partial t_0}(0,0) = 0$

which has the solution $u_0 = A_0(t_1)\cos(t_0)$ with $A_0(0) = a$.

Note that it is also possible to take $u_0 = A_0(t_1)\cos(t_0 + \phi(t_1)), A_0(0) = a, \phi(0) = 0$ but by expanding the cosine via $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, we see that the term proportional to $\sin(t_0)$ needs to be made zero so that the IC is satisfied (resulting in $\phi = 0$).

# Multiple Scales Expansion

- $\mathcal{O}(\varepsilon)$ $\quad \frac{\partial^2 u_1}{\partial t_0^2} + u_1 = -2\frac{\partial u_0}{\partial t_0} - 2\frac{\partial^2 u_0}{\partial t_1 \partial t_0}, \quad u_1(0,0) = 0, \left(\frac{\partial u_1}{\partial t_0} + \frac{\partial u_0}{\partial t_1}\right)|_{(0,0)} = 0$

The right hand side is found to be $f = 2(A_0(t_1) + A_0'(t_1))\sin(t_0)$.

Two approaches are possible now. Either this inhomogeneous ODE (in $t_0$) is solved with the right hand side $f$ and then $A_0$ is chosen so that secular terms (those growing in $t$) become zero or we invoke Fredholm alternative to get a condition on $f$ as the first step. We choose the latter approach.

Application of Fredholm alternative gives that for $L$ elliptic operator $Lu = f$ has a $2\pi-$periodic solution $u$ if and only if $f$ is $L^2(0,2\pi)$-orthogonal to $\text{Ker}(L^*)$, i.e.

$$\int_0^{2\pi} fv dt = 0 \quad \forall v \in \text{Ker}(L^*)$$

In our case $L = L^*$ and $\text{Ker}(L) = c_1\sin(t_0) + c_2\cos(t_0),\ c_j \in \mathbb{R}$ and we get the condition

$$\int_0^{2\pi} f\sin(t_0)dt_0 = \int_0^{2\pi} 2(A_0(t_1) + A_0'(t_1))\sin^2(t_0)dt_0 = 0$$

$$\Rightarrow \qquad A_0' + A_0 = 0$$

# Multiple Scales Expansion

Therefore $A_0 = ce^{-t_1}$, which together with the above condition $A_0(0) = a$ yields $\boxed{A_0 = ae^{-t_1}}$.
The $\mathcal{O}(\varepsilon)$ problem reduces to
$\frac{\partial^2 u_1}{\partial t_0^2} + u_1 = 0, \quad u_1(0,0) = 0, \frac{\partial u_1}{\partial t_0}(0,0) = -\frac{\partial u_0}{\partial t_1}(0,0) = a.$
Therefore $u_1 = A_1(t_1)\sin(t_0)$ with $A_1(0) = 0$.

To summarize we have

$$\boxed{u \sim ae^{-\varepsilon t}\cos(t) + \varepsilon A_1(\varepsilon t)\sin(t).}$$

To obtain $A_1$, we need to solve the $\mathcal{O}(\varepsilon^2)$ problem but even the first term in this asymptotic expansion gives a better approximation of the solution for time $\mathcal{O}(1)$ than two terms of the regular perturbation expansion.

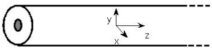
# Simplified NLS Derivation for Pulses in Optical Fibers

- for the full general derivation see [A. Newell and J.V. Moloney, “Nonlinear Optics,” Adison-Wesley, 1992.]

Maxwell equations ((1a) = Faraday’s law, (1b) = Ampere’s law):

$$\partial_t \vec{B} + \nabla \times \vec{E} = 0 \qquad (1a)$$

$$\nabla \times \vec{H} = \partial_t \vec{D} + \vec{J} \qquad (1b)$$




$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \quad \vec{B} = \mu_0 \vec{H} + \vec{M}, \quad \vec{P} = \epsilon_0 \left( \chi^{(1)}(x,y,z)\vec{E} + \chi^{(3)}(x,y,z)|\vec{E}|^2\vec{E} \right)$$

assumptions:

$$\vec{J} = 0 \quad \text{dielectric}, \qquad \vec{M} = 0, \qquad \nabla \cdot \vec{E} = 0 \quad \text{div. free field } \vec{E}$$

$\mu_0 \partial_t$(1b) $-\nabla \times$(1a): (use $\mu_0 \epsilon_0 = c^{-2}$)

$$-\nabla \times (\nabla \times \vec{E}) = \frac{1}{\epsilon_0 c^2} \partial_t^2 \vec{D}$$

$$\triangle \vec{E} - \nabla(\underbrace{\nabla \cdot \vec{E}}_{=0}) = \frac{1}{c^2} \partial_t^2 \left( \underbrace{\vec{E} + \chi^{(1)}\vec{E}}_{=:\, n_0^2 \vec{E}} + \chi^{(3)}|\vec{E}|^2\vec{E} \right)$$

($n_0$ ... lin. refractive index)

$$\triangle \vec{E} - \frac{n_0^2}{c^2}\partial_t^2 \vec{E} - \frac{1}{c^2}\partial_t^2\left(\chi^{(3)}|\vec{E}|^2\vec{E}\right) = 0$$

Further restrictions:

(a) $n_0 = n_0(x,y),\ \chi^{(3)} =$const.

(b) amplitude $|\vec{E}|$ s.t. nonlinearity and dispersion are same order effects

(c) polarization preserving fiber

(d) quasimonochromatic wavepackets around freq. $\omega_0$

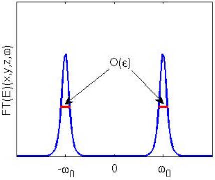
(e) monomode fiber

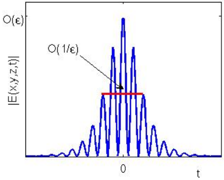
$$\text{(b)} \Rightarrow \vec{E} = \varepsilon \vec{E}_0 + \varepsilon^2 \vec{E}_1 + \varepsilon^3 \vec{E}_2 + \dots, \quad 0 < \varepsilon << 1$$

$$\text{(c)} \Rightarrow \vec{E}_k = E_k \vec{s}, \quad \vec{s} \quad \text{constant vector in } \mathbb{R}^3\text{, e.g.} \quad \vec{s} = (1,0,0)^T$$

(b,d)

$\Rightarrow$




i.e. $E_0 = U(x,y,\omega_0)\left[A(Z_1,Z_2,T)e^{i(k_0 z - \omega_0 t)} + \text{c.c.}\right], \quad Z_1 = \varepsilon z, Z_2 = \varepsilon^2 z, T = \varepsilon t$

$$\triangle E - \frac{n_0^2}{c^2}\partial_t^2 E - \frac{1}{c^2}\partial_t^2\left(\chi^{(3)}|E|^2 E\right) = 0, \qquad E \sim \varepsilon E_0 + \varepsilon^2 E_1 + \varepsilon^3 E_2$$

$$E_0 = U(x,y,\omega_0)\left[A(Z_1,Z_2,T)e^{i(k_0 z - \omega_0 t)} + \text{c.c.}\right], \qquad Z_1 = \varepsilon z, Z_2 = \varepsilon^2 z, T = \varepsilon t$$

$$E_{1,2} = E_{1,2}(x,y,z,t,Z_1,Z_2,T)$$

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Note: we replace $\partial_t \rightsquigarrow \partial_t + \varepsilon\partial_T,\ \partial_z \rightsquigarrow \partial_z + \varepsilon\partial_{Z_1} + \varepsilon^2\partial_{Z_2}$

- $\mathcal{O}(\varepsilon)$

$$Ae^{i(k_0 z - \omega_0 t)}\left(\triangle_{x,y} - k_0^2 + \frac{n_0^2\omega_0^2}{c^2}\right)U = 0 \qquad \text{and equivalent c.c. equation}$$

Obtain eigenvalue problem for $(k_0^2, U)$

$$\left(\triangle_{x,y} - k_0^2 + \frac{n_0^2\omega_0^2}{c^2}\right)U = 0$$

(e) $\Rightarrow$ only one eigenpair with $U \to 0$ as $x^2 + y^2 \to \infty$ exists

- $\mathcal{O}(\varepsilon^2)$

$$\left(\triangle - \frac{n_0^2}{c^2}\partial_t^2\right) E_1 + \left[2iUe^{i(k_0 z - \omega_0 t)}\left(k_0\partial_{Z_1} - \frac{n_0^2\omega_0}{c^2}\partial_T\right)A + \text{c.c.}\right] = 0$$

$$LE_1 = 2iUe^{i(k_0 z - \omega_0 t)}\left(\frac{n_0^2\omega_0}{c^2}\partial_T - k_0\partial_{Z_1}\right)A + \text{c.c.}, \quad L := \triangle - \frac{n_0^2}{c^2}\partial_t^2$$

Fredholm alternative: impose orthogonality of rhs to Ker($L^*$) (note: $L^* = L$)

orthog. satisfied except for $Ue^{i(k_0 z - \omega_0 t)} \in \text{Ker}(L) \quad \Rightarrow \quad \frac{n_0^2\omega_0}{c^2}\partial_T A - k_0\partial_{Z_1}A = 0$

$$\partial_{Z_1}A - \frac{n_0^2\omega_0}{c^2 k_0}\partial_T A = 0 \quad \Rightarrow \quad A \text{ travels at the velocity } \frac{n_0^2\omega_0}{c^2 k_0}$$

$$\therefore \quad \text{get } LE_1 = 0 \quad \Rightarrow \quad \text{take } E_1 = 0$$

**Claim:** $\frac{n_0^2\omega_0}{c^2k_0}$ is the group velocity at $\omega_0$, i.e. $\frac{n_0^2\omega_0}{c^2k_0} = k'(\omega_0)$

Pf.: $\left(\triangle_{x,y} + \frac{n_0^2\omega_0^2}{c^2} - k_0^2\right) U = 0 \quad \Rightarrow \quad -\langle \nabla U, \nabla U\rangle + \left(\frac{n_0^2\omega_0^2}{c^2} - k_0^2\right)\langle U, U\rangle = 0$

$$k_0^2 = \frac{n_0^2\omega_0^2}{c^2} - \mu, \qquad \mu := \frac{\langle \nabla U, \nabla U\rangle}{\langle U, U\rangle}$$

$$\Rightarrow \quad k_0 = \pm\left(\frac{n_0^2\omega_0^2}{c^2} - \mu\right)^{1/2}, \qquad k_0'(\omega_0) = \frac{n_0^2\omega_0}{c^2k_0} \qquad \square$$

Note: $k_0'' = \frac{n_0^2}{c^2}\left(\frac{1}{k_0} - \frac{\omega_0 k_0'}{k_0^2}\right) = \frac{n_0^2}{c^2}\left(\frac{1}{k_0} - \frac{n_0^2\omega_0^2}{c^2k_0^3}\right) = \frac{n_0^2}{c^2k_0}\left(1 - \frac{n_0^2\omega_0^2}{c^2k_0^2}\right)$

- $\mathcal{O}(\varepsilon^3)$ First calculate the nonlinearity: $|E_0|^2 E_0 =$

$$= U^3 \left[ \left( 2|A|^2 + A^2 e^{2i(k_0 z - \omega_0 t)} + {A^*}^2 e^{-2i(k_o z - \omega_0 t)} \right) \left( A e^{i(k_o z - \omega_0 t)} + A^* e^{-i(k_o z - \omega_0 t)} \right) \right]$$

$$= U^3 \left[ 3|A|^2 A e^{i(k_o z - \omega_0 t)} + A^3 e^{3i(k_o z - \omega_0 t)} + \text{c.c.} \right]$$

Thus

$$LE_2 = \left( -\partial_{Z_1}^2 A - 2ik_0 \partial_{Z_2} A + \frac{n_0^2}{c^2} \partial_T^2 A \right) U e^{i(k_0 z - \omega_0 t)}$$

$$- \frac{\omega_0^2}{c^2} \left( 3|A|^2 A e^{i(k_0 z - \omega_0 t)} + A^3 e^{3i(k_0 z - \omega_0 t)} \right) U^3 + \text{c.c.}$$

Orthogonalize rhs to Ker($L$): (term with $e^{\pm 3i(k_0 z - \omega_0 t)}$ already orthog.)

$$i\partial_{Z_2} A + \frac{1}{2k_0} \left( \partial_{Z_1}^2 A - \frac{n_0^2}{c^2} \partial_T^2 A + 3 \frac{\langle U^3, U \rangle}{\langle U, U \rangle} \frac{\omega_0^2}{c^2} |A|^2 A \right) = 0$$

Let $\tau := \frac{n_0^2 \omega_0}{c^2 k_0} Z_1 + T$ $\left( \partial_{Z_1} = \frac{n_0^2 \omega_0}{c^2 k_0} \partial_\tau, \partial_T = \partial_\tau \right)$, $\rho := \frac{\langle U^3, U \rangle}{\langle U, U \rangle}$

$$i\partial_{Z_2} A - \frac{1}{2} \frac{n_0^2}{c^2 k_0} \left( 1 - \frac{n_0^2 \omega_0^2}{c^2 k_0^2} \right) \partial_\tau^2 A + \frac{3}{2k_0} \frac{\omega_0^2}{c^2} \rho |A|^2 A = 0$$

$$\boxed{i\partial_{Z_2} A - \frac{k_0''}{2} \partial_\tau^2 A + \frac{3}{2k_0} \frac{\omega_0^2}{c^2} \rho |A|^2 A = 0}$$