Surface Gap Solitons at a Nonlinearity Interface

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Outline

Introduction

Surface wave = wave inherent to the interface of 2 (or more) materials

Examples of nonlinear electromagnetic surface waves

- surface plasmon polaritons at the metal/dielectric interface [Kawata, 2001]
- interface of a linear and Kerr nonlinear dielectric [Maradudin, 1983; Tomlinson, 1980]
- interface of two homogenous Kerr nonlinear dielectrics [Maradudin, 1983; Stegeman et al, 1985; Boardman et al, 1991]
- interface of a periodic lattice and a homogenous dielectric [Makris et al., 2005],[EXPERIMENT:Suntsov et al., 2006]
- interface of two periodic dielectrics with saturable nonlinearity and with different $\bar{n}_0$ [Kartashov et al., 2006]

Our setting:

- interface of two periodic media with different coefficients of Kerr nonlinearity
  - periodicity $\Rightarrow$ gaps in the linear spectrum
  - nonlinear (de)focusing $\Rightarrow$ solitary waves in the gaps possible (gap solitons)
  - material interface $\Rightarrow$ special solitary waves localized at the interface

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Introduction - Mathematical Model

Gross-Pitaevsky equation

$$iu_t = -u_{xx} + V(x)u - \Gamma(x)|u|^2 u, \qquad x \in \mathbb{R}, t \geq 0$$
$$\Gamma(x) = \Gamma_+ \text{ for } x \geq 0, \Gamma_- \text{ for } x < 0$$

$$V(x)\ldots \text{real, continuous and} \begin{cases} \text{case } \mathbf{A}: \; V(x) \text{ smooth, } d\text{-periodic} \\ \text{case } \mathbf{B}: V(x) \; d\text{-periodic on } x \geq 0 \text{ and on } x < 0 \\ \qquad\qquad V'(x) \text{ discontin. at } x = 0, \end{cases}$$

Choose: $V_0(x) = \sin^2\left(\frac{\pi x}{10}\right)$

case **A**: $V(x) = V_0(x)$

case **B**: $V(x) = V_0(x-\delta)\chi_{(-\infty,0)} + V_0(x+\delta)\chi_{[0,\infty)}$

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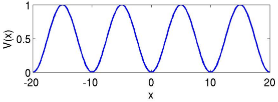

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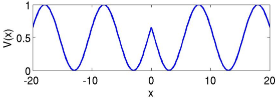

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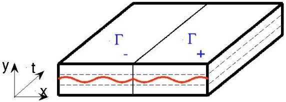
Applications:

- beam propagation in Kerr nonlinear dielectrics with transverse periodicity



- Bose-Einstein Condensates loaded on an optical lattice

Note: In most dielectrics a jump in the nonlinear part of n $\Rightarrow$ much larger jump in the linear part

BUT not in all materials [Blömer et al., 2006]

Aim: single-hump, exponentially decaying, solitary wave solutions $u = e^{-i\omega t}\phi(x), \phi : \mathbb{R} \to \mathbb{R},\ \phi \in C^1(\mathbb{R})$ centered at $x = 0$

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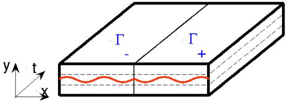

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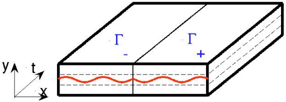

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Linear problem for a smooth $V(x)$ (Floquet theory)

$$L\psi = \omega\psi, \qquad L = -\partial_x^2 + V(x), V(x+d) = V(x)$$

Hill's equation:

$\sigma(L) = \sigma_c(L) = \Sigma = [\omega_0, \omega_1] \cup [\omega_2, \omega_3] \cup \dots$

$\omega_{2n-2} < \omega_{2n-1} \leq \omega_{2n}, \quad \omega_n \to \infty$ as $n \to \infty$

- $\omega \in \text{int}(\Sigma) \quad \Rightarrow \quad \psi$ quasi-periodic:
  $\psi = p(x)e^{\pm ikx}, k \in [0, \pi/d], p(x+d) = p(x)$
- $\omega = \omega_n \quad \Rightarrow$
  one solution $\psi_n(x)$ periodic or anti-periodic ($k = 0$ or $k = \pi/d$)
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n	0	1	2	3	4	5	6	7
symmetry	even	even	odd	odd	even	even	odd	odd
periodicity	S_+	S_-	S_-	S_+	S_+	S_-	S_-	S_+
sign($\omega''(k)$)	1	-1	1	-1	1	-1	1	-1

$S_+ \dots d-$periodic, $\qquad S_- \dots d-$antiperiodic

[Eastham, 1973]

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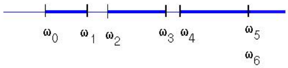

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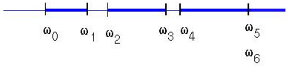

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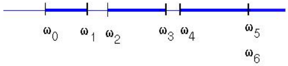

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$$-\phi'' - \omega\phi + V(x)\phi - \Gamma_0\phi^3 = 0, \quad x \in \mathbb{R}$$

Gap solitons exist for all $\omega \in \mathbb{R} \setminus \Sigma$. [Pelinovsky et al., 2004; Pankov, 2005]

- $\omega'' > 0$ at even band edges $\Rightarrow$ GS's bifurcate from ω_{2m} for $\Gamma_0 > 0$
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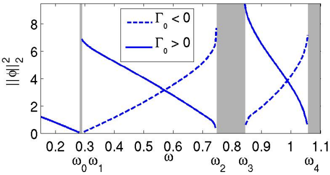

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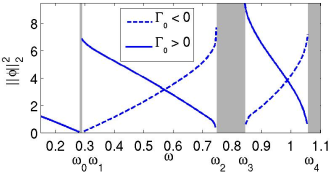

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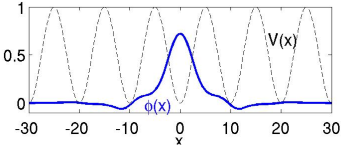

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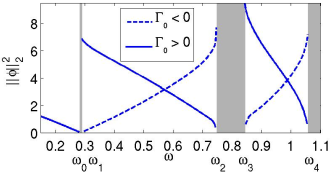

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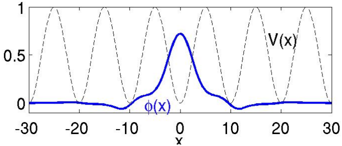

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2. Background
   - Floquet Theory
   - Gap Solitons
3. Surface Gap Solitons - Smooth Potential (A)
   - Bifurcations of SGS's from Nondegerate GS's
   - Numerical Computations
   - Asymptotics near GS Bifurcation Points
4. Surface Gap Solitons - per. potential with singularity at $x = 0$ (B)
   - Gluing Technique
   - Numerical Homotopy Continuation
5. Conclusions, Future Work

SGS's for a Smooth Potential $V(x)$ and $\Gamma_+ \neq \Gamma_-$

Bifurcations of SGS's from GS's:

Rewrite $-\phi'' - \omega\phi + V(x)\phi - \Gamma(x)\phi^3 = 0$ as

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- $J = D_\phi F(\phi_{(0,\gamma_0)}, 0, \gamma_0) = -\partial_x^2 - \omega + V(x) - 3\gamma_0\phi^2_{(0,\gamma_0)}$
- J ... relatively compact perturb. of $L - \omega$

$$\Rightarrow \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(L - \omega) = \Sigma - \omega$$

- $\omega \in \mathbb{R} \setminus \Sigma \Rightarrow d(0, \sigma_{\text{ess}}(J)) > 0.$
- *nondegeneracy condition:* no zero e-vals of J (expected as shift invariance is broken)

$\therefore$ Impl. Function Thm: $\exists! \; \phi_{(\nu,\gamma)}$ for (ν, γ) suff. close to $(0, \gamma_0)$

Conclusion: If J is nonsingular, a nondegenerate GS can be continued to a SGS family for sufficiently small $|\Gamma_+ - \Gamma_-|$.

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- $J \ldots$ relatively compact perturb. of $L - \omega$

$$\Rightarrow \quad \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(L - \omega) = \Sigma - \omega$$

- $\omega \in \mathbb{R} \setminus \Sigma \;\Rightarrow\; d(0, \sigma_{\text{ess}}(J)) > 0.$
- *nondegeneracy condition:* no zero e-vals of J (expected as shift invariance is broken)

$\therefore$ Impl. Function Thm: $\exists! \; \phi_{(\nu,\gamma)}$ for (ν, γ) suff. close to $(0, \gamma_0)$

Conclusion: If J is nonsingular, a nondegenerate GS can be continued to a SGS family for sufficiently small $|\Gamma_+ - \Gamma_-|$.

SGS's for a Smooth Potential $V(x)$ and $\Gamma_+ \neq \Gamma_-$

Numerical results

- homotopy continuation from a GS via Newton iteration on a centered FD discretization
  - Γ_+ fixed (=1 or -1) and Γ_- deviated from Γ_+ up to a threshold $\Gamma_- = \Gamma_*$, where the convergence fails

Note: $|\Gamma_* - \Gamma_+| \to 0$ as $\omega \to \omega_n$, where ω_n is the local bifurc. point of the GS.

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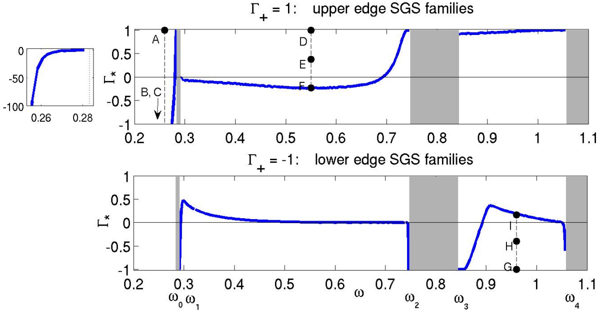

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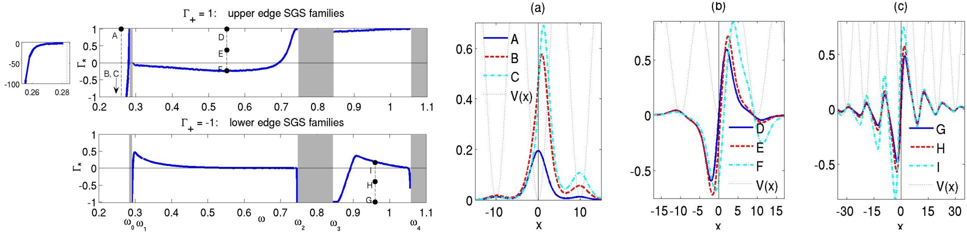

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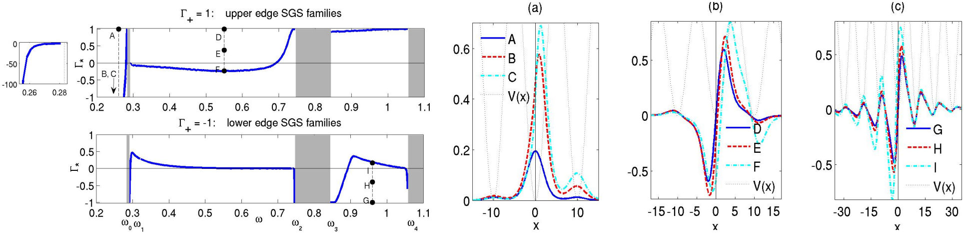

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Asymptotics near GS Bifurcation Points ω_n

For ω near ω_n GP is approximated by NLS:

$$\omega = \omega_n + \varepsilon^2 \Omega + \mathcal{O}(\varepsilon^4), \qquad \varepsilon << 1$$
$$\phi(x) = \varepsilon A(X)\psi_n(x) + \varepsilon^2 A'(X)\tilde{\psi}_n(x) + \mathcal{O}(\varepsilon^3), \qquad X = \varepsilon x$$

- ψ_n ...(anti)periodic Bloch function: $(L - \omega_n)\psi_n = 0$
- $\tilde{\psi}_n$ ...generalized (anti)periodic Bloch function: $(L - \omega_n)\tilde{\psi}_n = 2\psi_n'$

... multiple scales ...

$$\Omega A + \mu A'' + \rho\Gamma(X)A^3 = 0, \qquad \mu = 1 + 2\frac{(\tilde{\psi}_n, \psi_n)}{(\psi_n, \psi_n)}, \quad \rho = \frac{(\psi_n^2, \psi_n^2)}{(\psi_n, \psi_n)} \tag{2}$$

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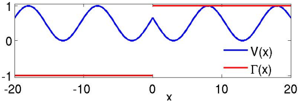
Outline

SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

$$-\phi'' - \omega\phi + V(x)\phi - \Gamma(x)\phi^3 = 0, \qquad \Gamma(x) = \Gamma_\pm \text{ for } x \geq 0, x < 0 \text{ resp.}$$

Set

- $\Gamma_+ = -\Gamma_- = 1$
- $V(x) = V_0(x-\delta)\chi_{(-\infty,0)} + V_0(x+\delta)\chi_{[0,\infty)}$



Q: Do SGS's have anything in common wit GS's on each half line?
A: Gluing algorithm to generate SGS's:

-step 1- continuous solutions

Let $\omega \in (\omega_{2m-1}, \omega_{2m})$.

$\phi_+(x;\omega)\ldots$ GS centered at $x = 0$ with $\Gamma \equiv \Gamma_+, V = V_0$
$\phi_-(x;\omega)\ldots$ GS centered at $x = 0$ with $\Gamma \equiv \Gamma_-, V = V_0$

Fix $\delta \in (0, d)$ and define:

$$f_A(\omega) = \phi_-(-\delta;\omega) - \phi_+(\delta;\omega)$$
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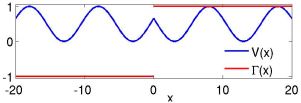

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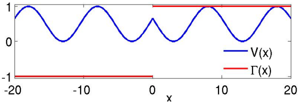

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$\forall \delta > 0 \quad \phi_-(-\delta,\omega)$ must intersect $\phi_+(\delta,\omega)$ or $-\phi_+(\delta,\omega)$ in $(\omega_{2m-1}, \omega_{2m})$ since

- $\phi_\pm$ are continuous in ω
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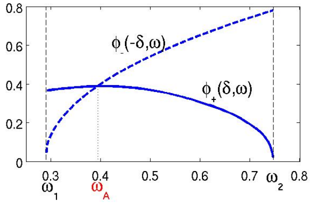

Obtain δ-parametrized family of continuous solutions $\phi_{A,B}$

$$\phi_A(x;\delta) = \phi_-(x-\delta;\omega_A)\chi_{(-\infty,0)} + \phi_+(x+\delta;\omega_A)\chi_{[0,\infty)}$$
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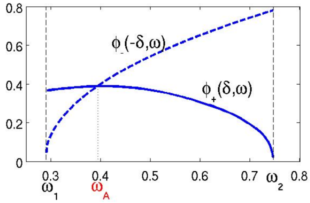

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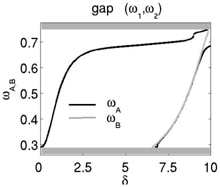

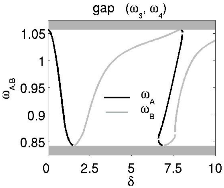

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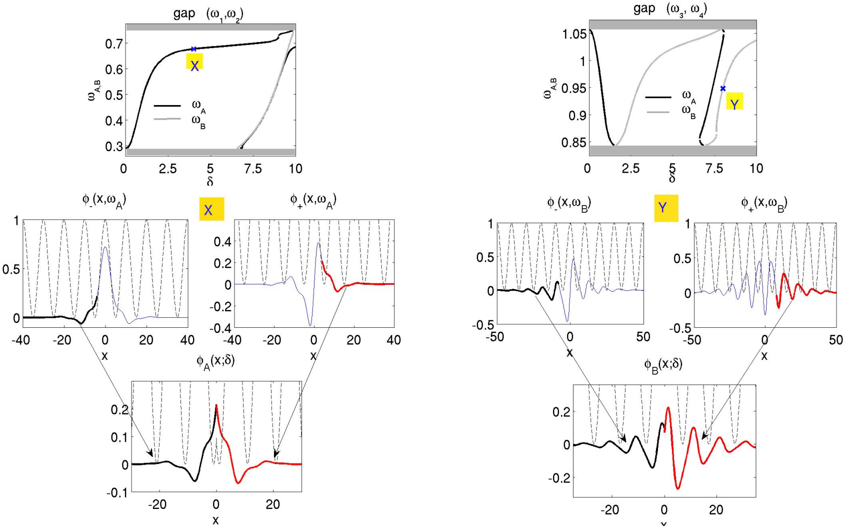

SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

-step 2- C^1 solutions

Search the family $\phi_{A,B}(x;\delta)$ for $C^1(\mathbb{R})$ solutions

Define
$$\begin{aligned} g_A(\delta) &:= \phi'_-(-\delta;\omega_A) - \phi'_+(\delta;\omega_A) \\ g_B(\delta) &:= \phi'_-(-\delta;\omega_B) + \phi'_+(\delta;\omega_B) \end{aligned}$$

δ_* ...zeros of either g_A or g_B

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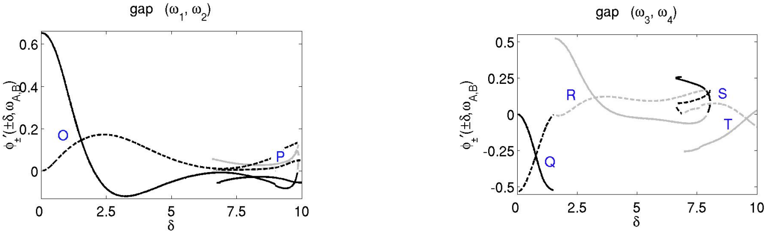

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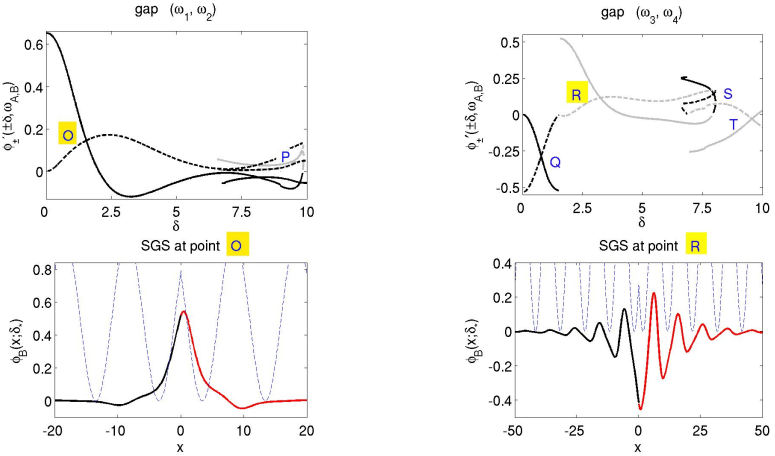

Homotopy Continuation of C^1 SGSs in ω

- SGS's at $O - T$ used as starting points of a homotopy continuation in ω:

- termination points $O_L - T_L$ are bound states of the linear e-val problem:

$$-\psi'' + V(x;\delta)\psi = \omega\psi, \qquad V(x;\delta) = V_0(x-\delta)\chi_{(-\infty,0)} + V_0(x+\delta)\chi_{[0,\infty)}$$

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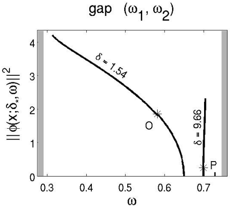

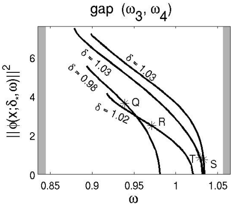

- termination points $O_L - T_L$ are bound states of the linear e-val problem:

$$-\psi'' + V(x;\delta)\psi = \omega\psi, \qquad V(x;\delta) = V_0(x-\delta)\chi_{(-\infty,0)} + V_0(x+\delta)\chi_{[0,\infty)}$$

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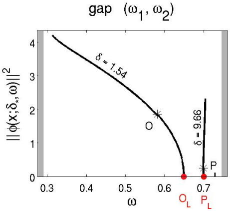

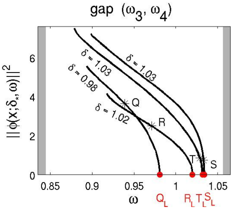

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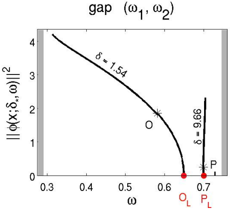

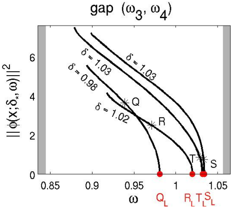

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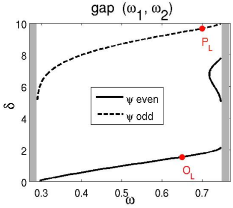
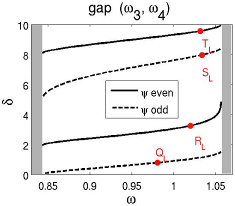


$$\sigma_p(-\partial_x^2 + V(x;\delta))$$

for $\delta \in (0, d)$



Outline

1. Introduction
2. Background
   - Floquet Theory
   - Gap Solitons
3. Surface Gap Solitons - Smooth Potential (A)
   - Bifurcations of SGS's from Nondegerate GS's
   - Numerical Computations
   - Asymptotics near GS Bifurcation Points
4. Surface Gap Solitons - per. potential with singularity at $x = 0$ (B)
   - Gluing Technique
   - Numerical Homotopy Continuation
5. **Conclusions, Future Work**

Conclusions, Future Work

Conclusions:

- 1D SGS at a nonlinearity interface in GP eq. - smooth per. potential $V(x)$
  - their bifurcation from nondegenerate GS's showed asymptotically
  - threshold nonlinearity jump computed
  - (threshold jump) $\rightarrow 0$ as $\omega \rightarrow$ (GS bifurcation edge)
- 1D SGS at a nonlinearity interface in GP eq. - jump in $V'(x)$ at $x = 0$
  - countably many SGS's found via gluing of 2 GS segments
  - homotopically continued into families within the frequency gaps
  - family termination points related to linear point spectrum

Future Work:

- nonlinearity interface with discontinuous $V(x)$ and $V'(x)$ at $x = 0$
- 2D, 3D: $-\nabla^2\phi - \omega\phi + V(\vec{x})\phi - \Gamma(\vec{x})\phi^3 = 0, \quad \Gamma = \Gamma_\pm$ for $x_1 \geq 0, x_1 < 0$
  - both smooth and non-smooth $V(\vec{x})$ possible
  - gluing technique not feasible - use homotopy from linear bound states instead
- linear stability of SGS's
  - inheritance of GS stability near bifurcation points expected
  - far from bifurcation points - possible stabilization due to the interface

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Bibliography I

M.S. Eastham
The Spectral Theory of Periodic Differential Equations.
Scottish Academic Press, 1973.

S. Kawata (ed.)
Near-Field Optics and Surface Plasmon Polaritons.
Springer, New York, 2001.

W. J. Tomlinson
Surface wave at a nonlinear interface
Opt. Lett., 5:323–325, 1980.

A.A. Maradudin
Nonlinear surface electromagnetic waves
Optical and Acoustic Waves in Solids - Modern Topics, M. Borissov, ed. 72–142, 1983.

G.I. Stegeman, C.T. Seaton, J. Ariyasu, R.F. Wallis and A.A. Maradudin
Nonlinear electromagnetic waves guided by a single interface
J. App. Phys., 58: 2453–2459, 1985.

A.D. Boardman, P. Egan, F. Lederer, U. Langbein and M. Mihalache
Third-order nonlinear electromagnetic TE and TM guided waves
Nonlinear Surface Electromagnetic Phenomena, Vol. 29 of *Modern Problems in Condensed Matter Science*, H.E. Ponath and G.I. Stegeman, eds., 1991.

D.E. Pelinovsky, A.A. Sukhorukov and Y. Kivshar
Bifurcations and stability of gap solitons in periodic structures
Physical Review E, 70(036618):1–14, 2004.

A. Pankov
Periodic nonlinear Schrödinger equation with application to photonic crystals
Milan J. Math., 73: 259–287, 2005.

Y.V. Kartashov, V.A. Vysloukh and L. Torner
Surface gap solitons
Phys. Rev. Lett., 96: 073901, 2006.

K.G. Makris, S. Suntsov, D.N. Christodoulides, G.I. Stegeman, and A. Hache
Discrete surface solitons
Opt. Lett., 30: 2466–2468, 2005.

D. Blömer, A. Szameit, F. Dreisow, T. Schreiber, S. Nolte, and A. Tünnermann
Nonlinear refractive index of fs-laser-written waveguides in fused silica
Opt. Express, 14: 2151-2157, 2006.

S. Suntsov, K. G. Makris, D. N. Christodoulides, G. I. Stegeman, A. Hache, R. Morandotti, H. Yang, G. Salamo, and M. Sorel
Observation of Discrete Surface Solitons
Phys. Rev. Lett., 96: 063901, 2006.