Transcritical flow over a step

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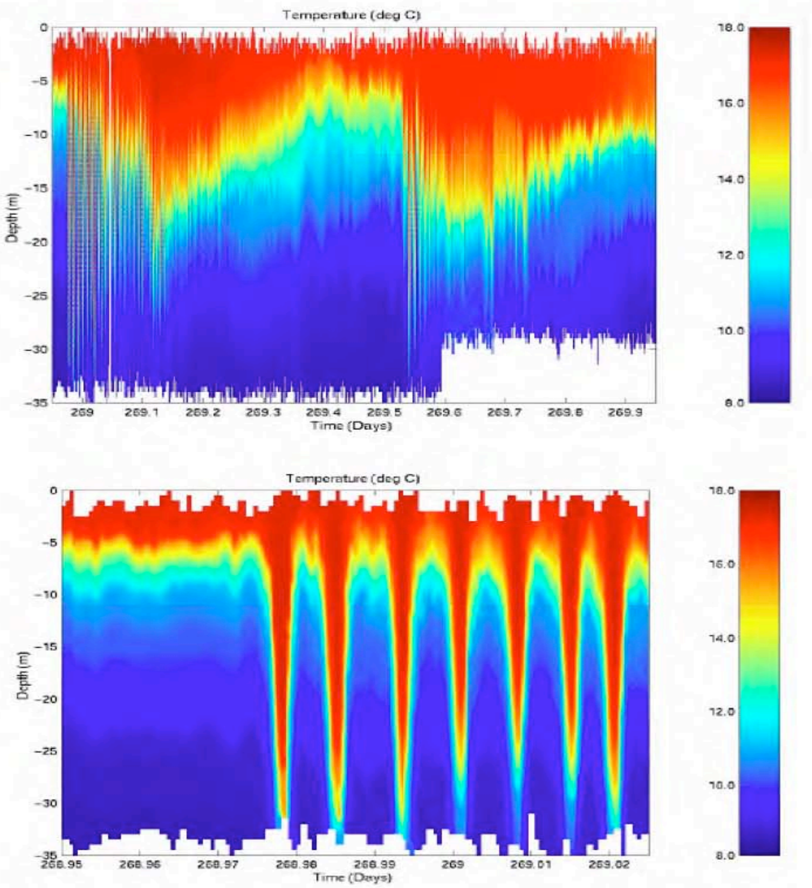
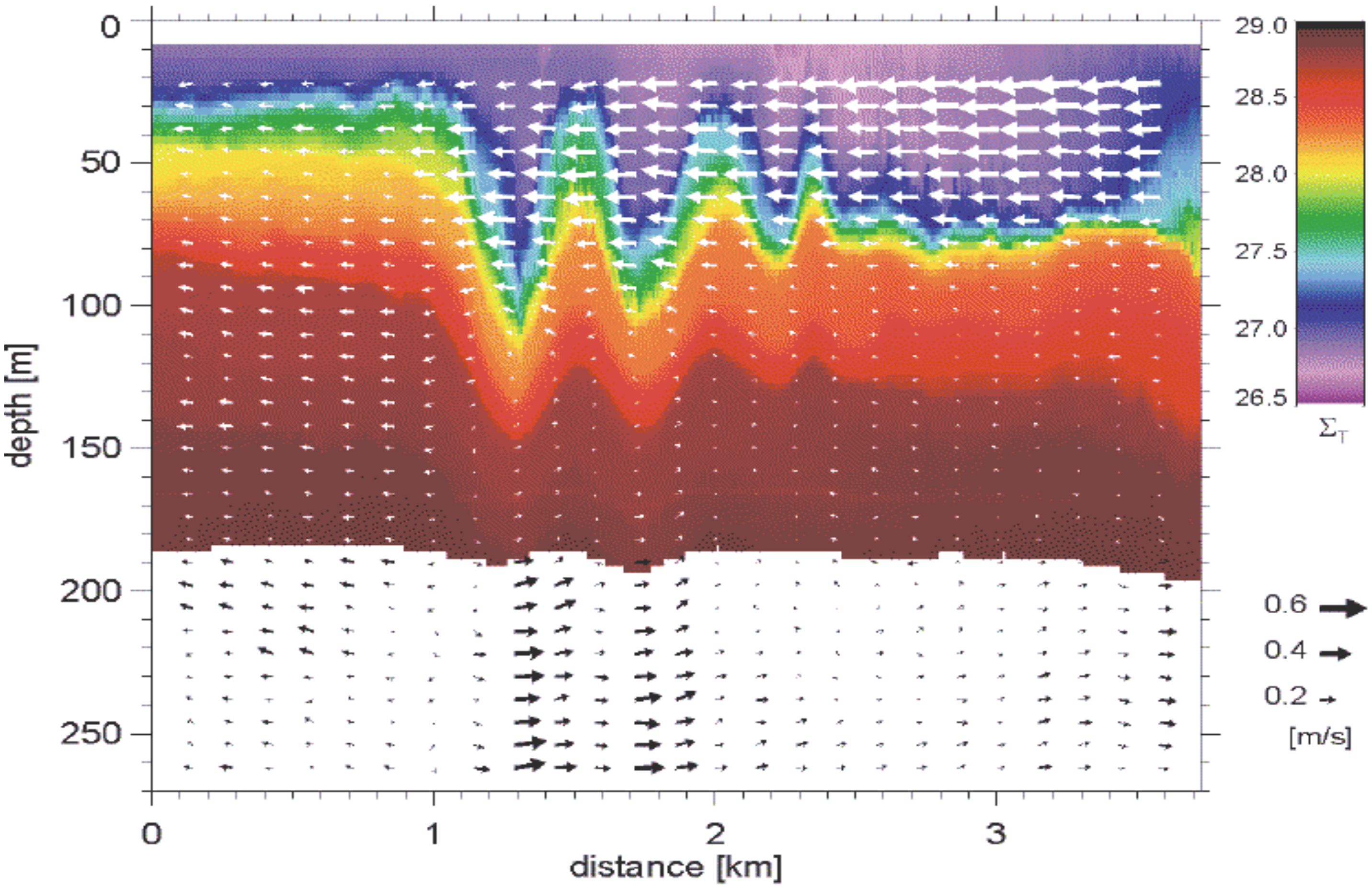
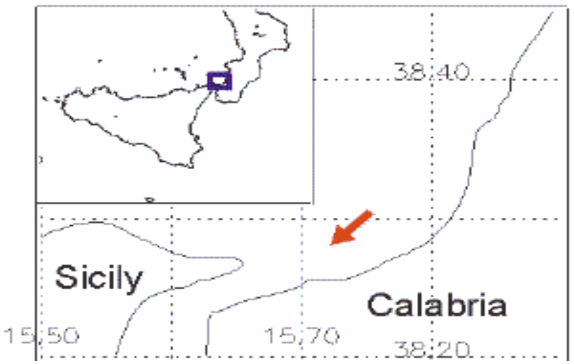


Figure 2. (Upper) A color contour time series of temperature profiles from the surface to 35m depth measured by the LMP over a one-day period. The 10°C span color contour scale is shown the right of the time series panel. The low frequency, semidiurnal internal tide displacement can clearly be seen along the yellow isotherm. (Lower) A profile time series of the first 1.7 hours of the time series shown in Figure 1a. White areas indicate times with no data. [From Stanton and Ostrovsky, 1998]

STRAIT OF MESSINA
October 25, 1995





start time: 15:50 UTC
position: 38.305 N 15.752 E

end time: 16:14 UTC
position: 38.281 N 15.722 E

ship speed (m/s): 2.5
ship heading (degrees): 225

MAX. NORTHWARD TIDAL FLOW: 16:02 UTC

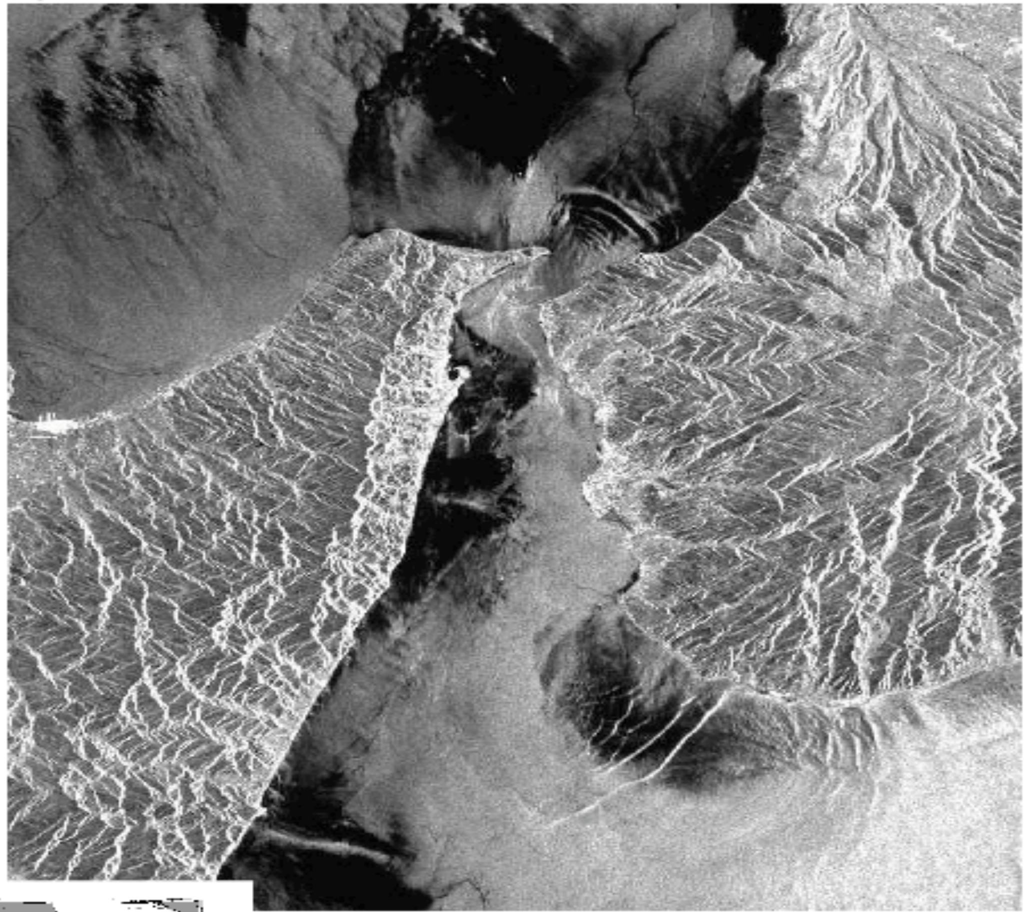
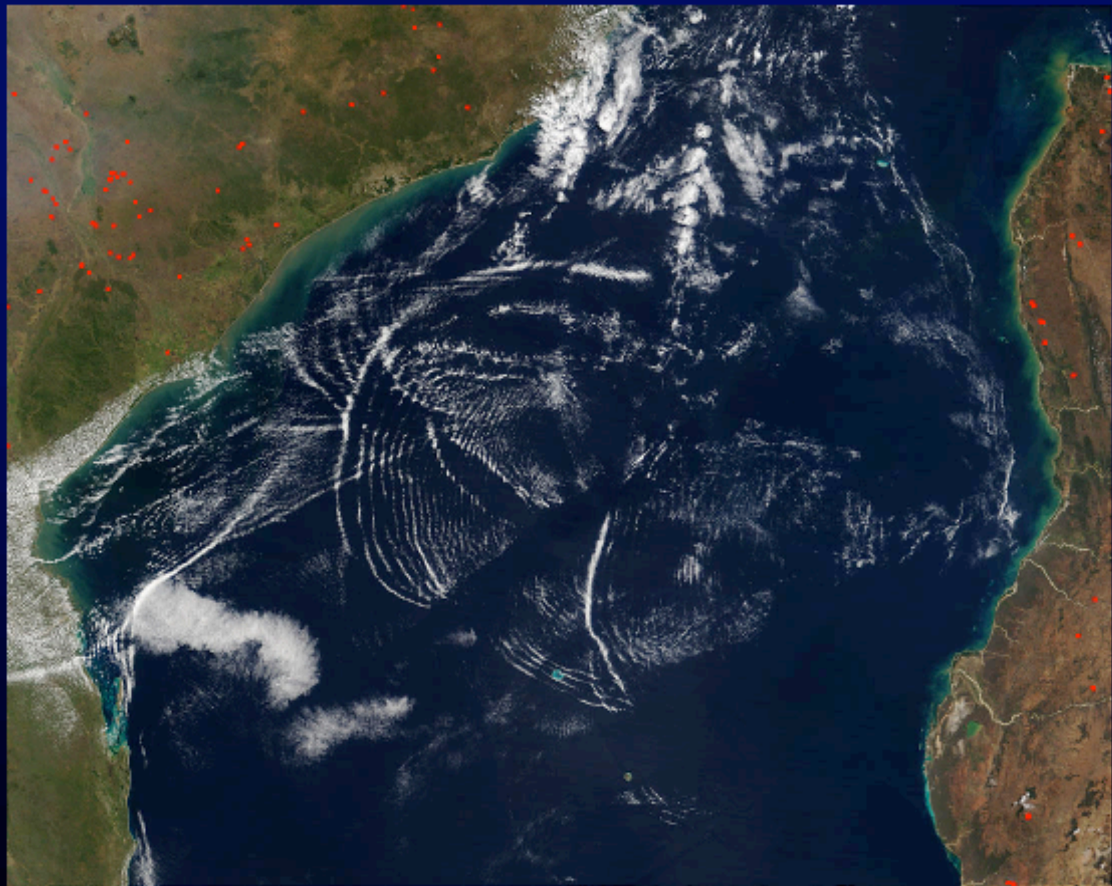


Figure 5. ERS-1 (C-band VV) SAR image of the Strait of Messina acquired on 11 July 1993 at 0941 UTC (orbit 10387, frame 2835). The image shows internal wave signatures radiating out of the strait in both the northern and southern directions. Northwards propagating internal waves are less frequently observed than southward propagating ones. Imaged area is 65 km x 65 km. ©ESA 1993. [From The Tropical and Subtropical Ocean Viewed by ERS SAR http://www.ifm.uni-hamburg.de/ers-sar/]

Morning Glory Waves of the Gulf of Carpentaria

Atmospheric internal waves



MODIS TERRA image Mozambique channel (16 AUG 2002)

Generation Mechanisms

- Interaction of the large-scale barotropic tide with the continental slope, which generates a meso-scale internal tide. As it propagates shoreward (or off-shore) the internal tide steepens and a soliton wave-train forms, the undular bore: Modeled by the KdV equation

- Critical flow (internal Froude number F, the ratio of current speed to linear long wave speed, is close to one) over a sill, or through a strait, generating upstream and downstream propagating undular bores. Modeled by forced KdV equation

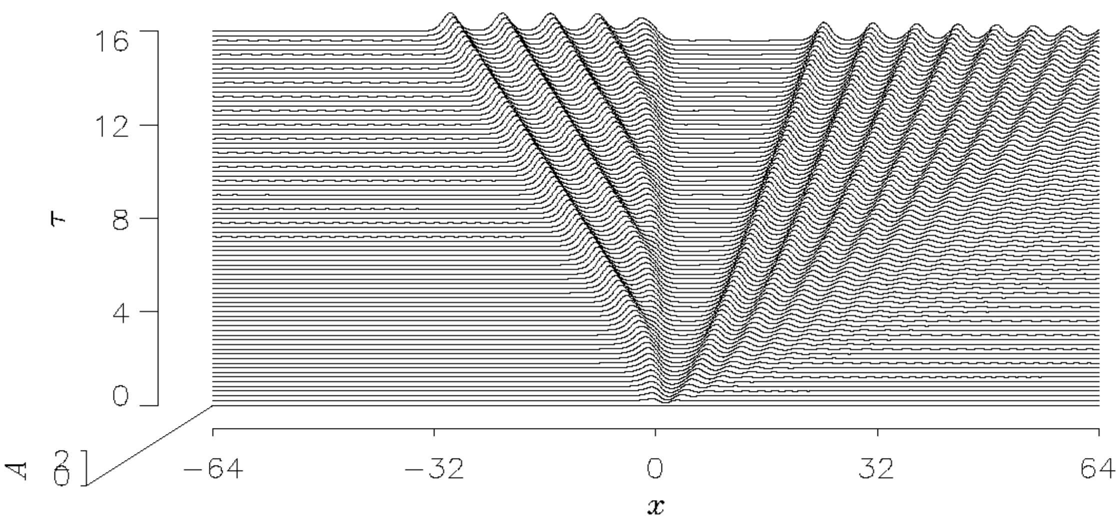
Forced KdV (fKdV) equation

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + G_x(x) = 0$$

Here A(x, t) is the wave amplitude, $\Delta = F - 1$ measures the degree of criticality, and G(x) is the projection of the topography onto the relevant (critical) mode.

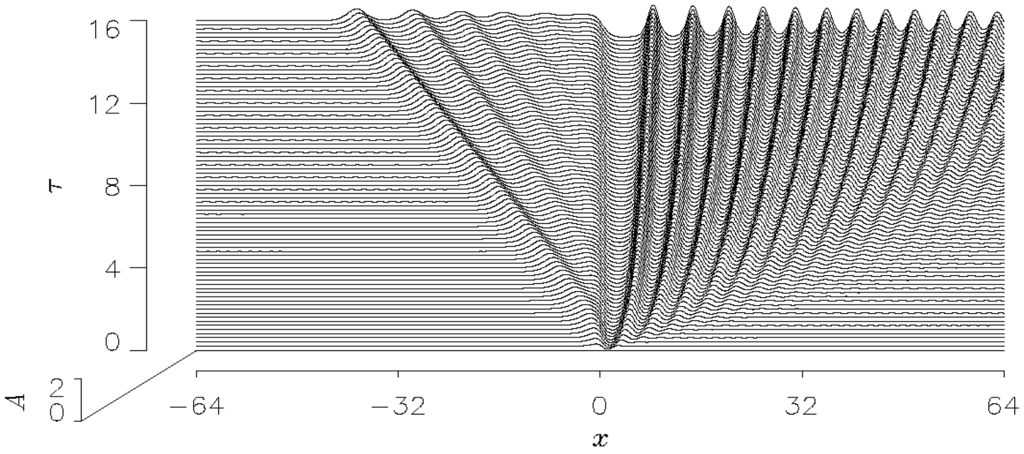
The equation is in canonical form, for an oncoming flow in the positive x-direction. It has been derived for both surface and internal waves.

Simulation at exact criticality, Δ=0, and for positive localized forcing of amplitude 1.



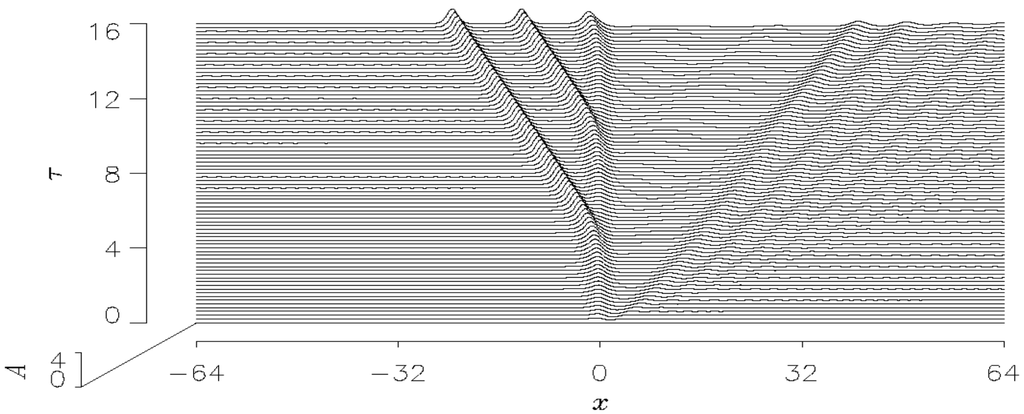
Δ=-1.5

subcritical



Δ=1.5

supercritical



These results are for a **localized** forcing function $G(x) = G_m \mathrm{sech}^2(x/\lambda)$, where the key parameter is the forcing height G_m (=1 in these plots, also $\lambda=1$).

Our interest here is when the forcing function is a single **step**, say $G(x) = G_m\{1+\tanh(x/\lambda)\}/2$, representing a step of height G_m (up when > 0).

However, since a single step causes a loss of mass in the fKdV equation, it is convenient to consider a step up and a step down simultaneously:

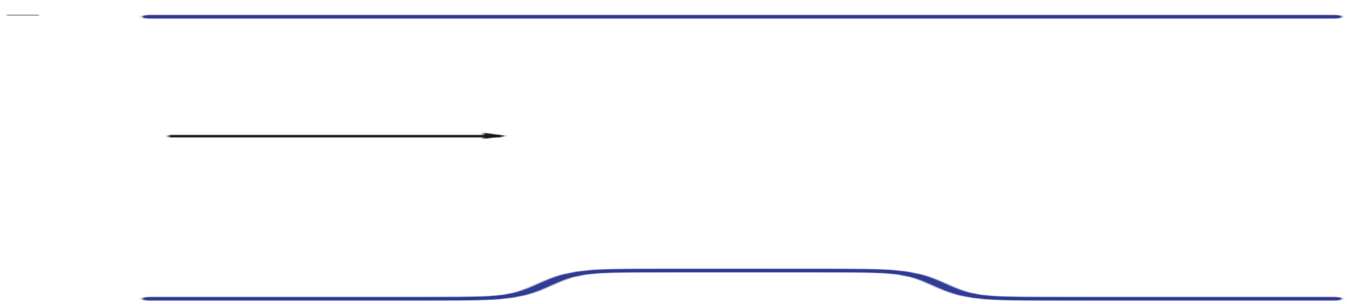
$$G(x) = G_m\{\tanh(x/\lambda)-\tanh(x-L/\lambda)\}/2\ , \quad \lambda \ll L,$$

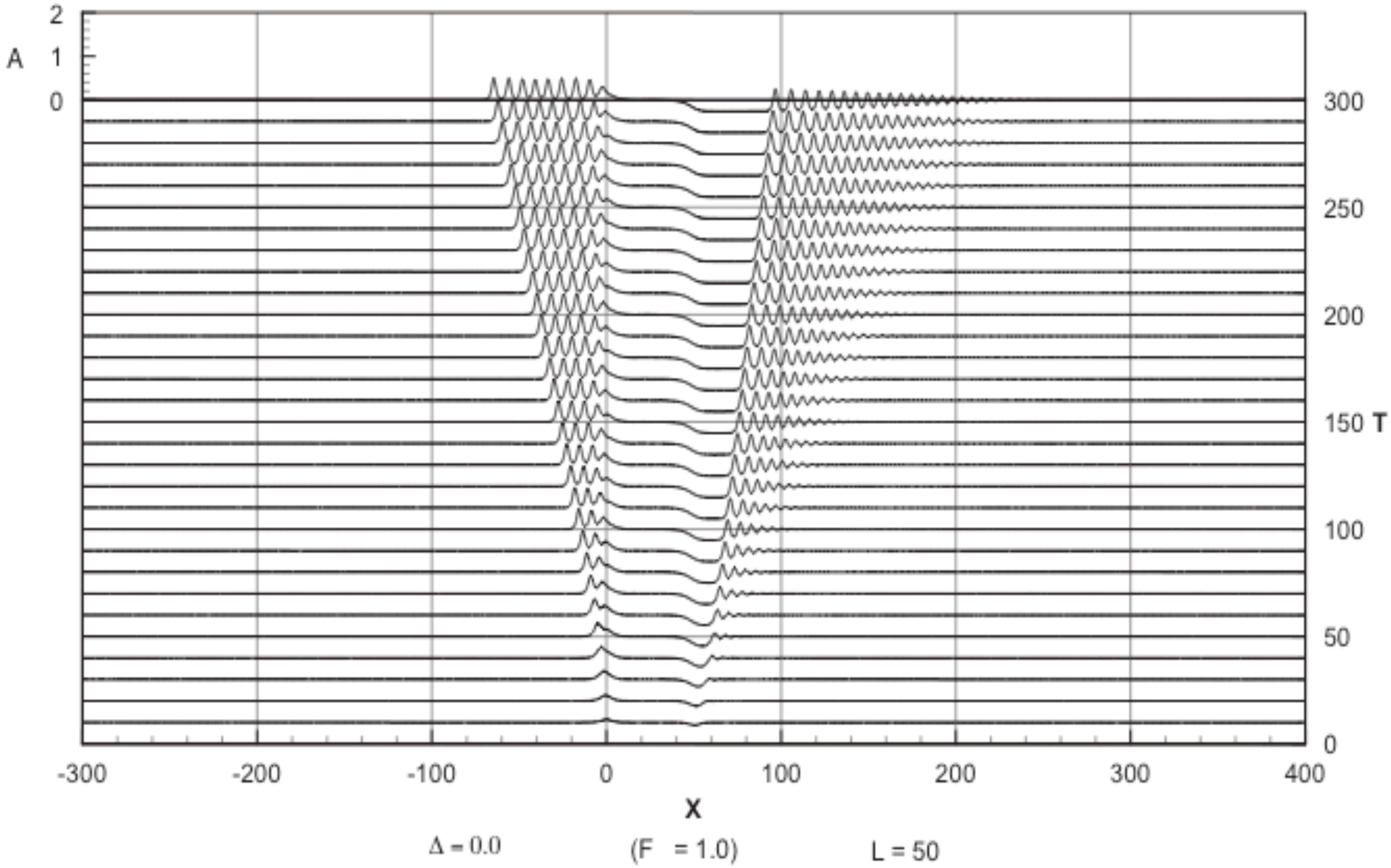
which is a step up (down) at x=0 (x=L) when $G_m > 0$.

Next we show plots for the fKdV equation:

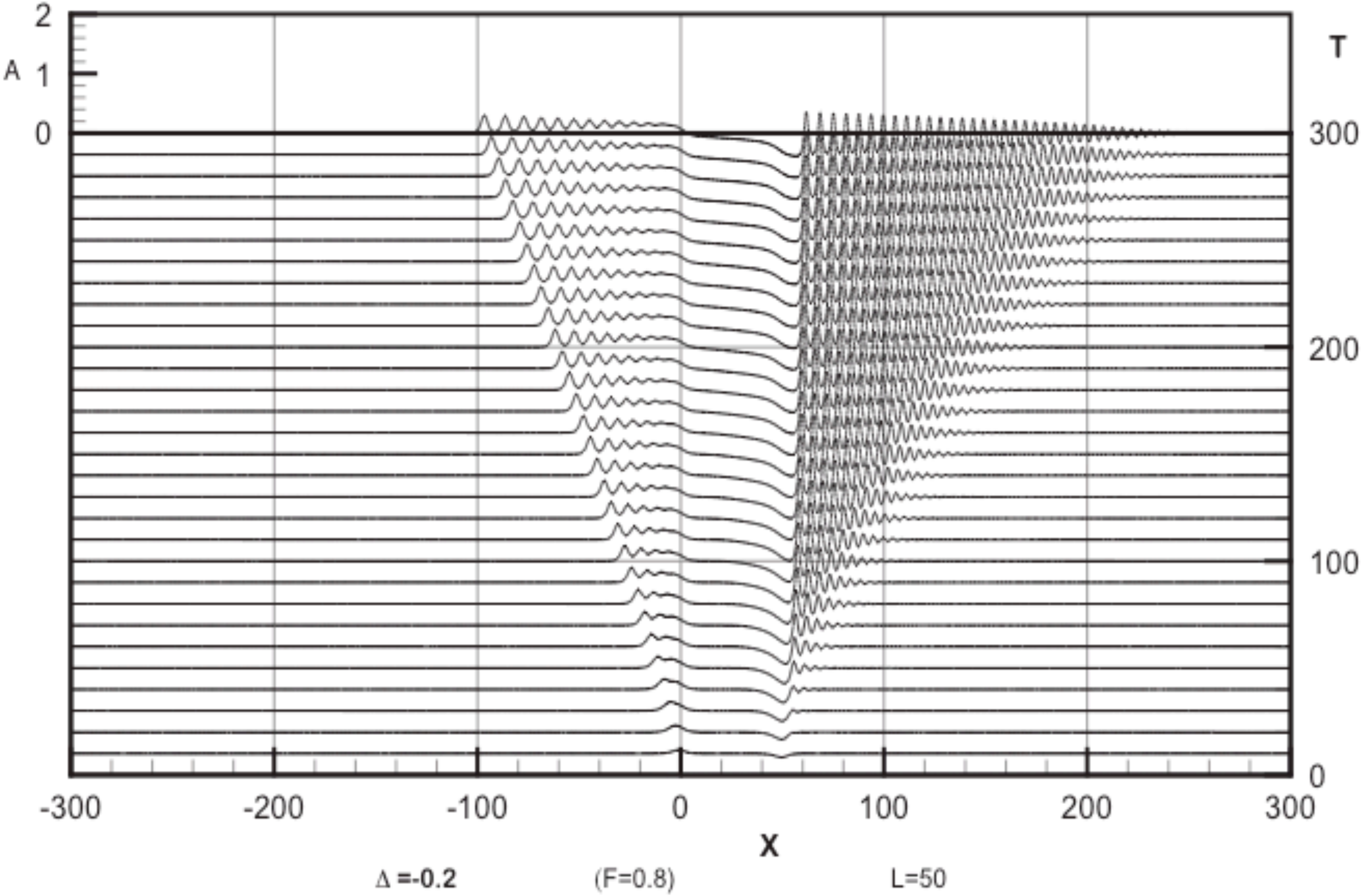
$$-A_t - \Delta A_x + 3/2AA_x + 1/6A_{xxx} + G_x(x) = 0$$

which is the scaled version appropriate for surface waves. Here $G_m=0.1$, $\lambda=4$, $L=50$, $\Delta=0, -0.2, 0.2$

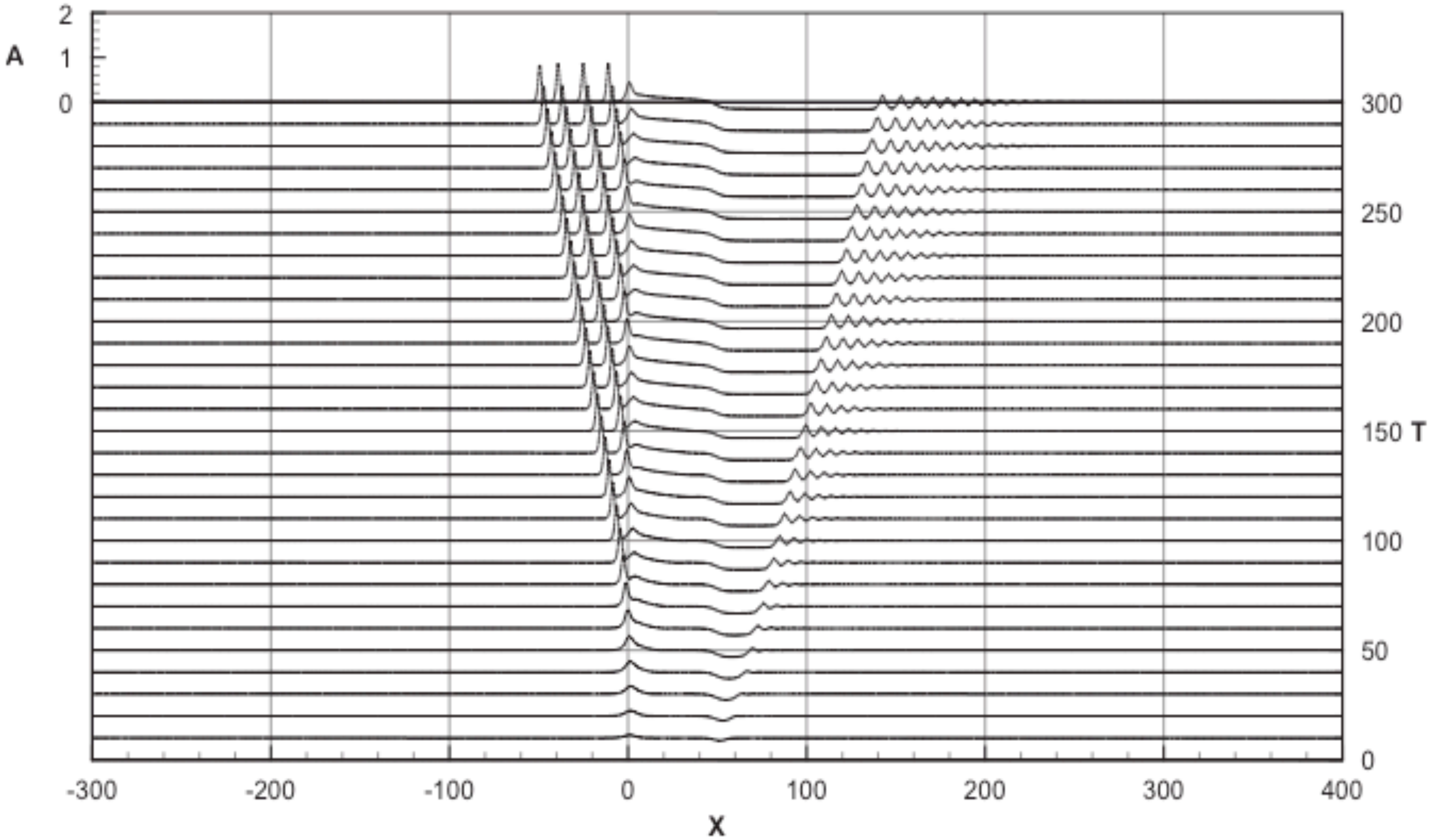




A
2
1
0
300
250
200
150 T
100
50
0
-300
-200
-100
0
100
200
300
400
X
Δ = 0.0
(F = 1.0)
L = 50

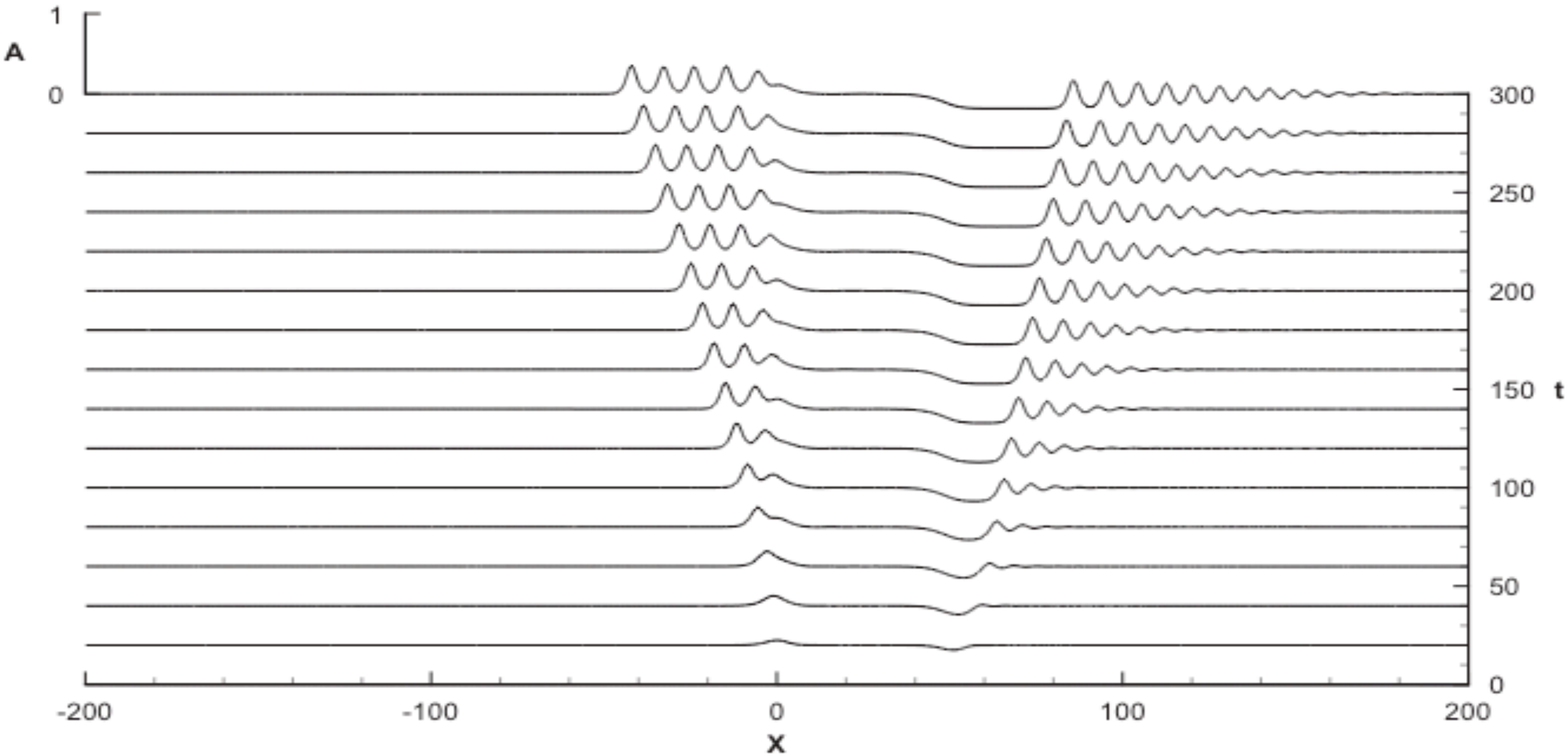


2
A 1
0
T
300
200
100
0
-300
-200
-100
0
100
200
300
X
Δ =-0.2
(F=0.8)
L=50

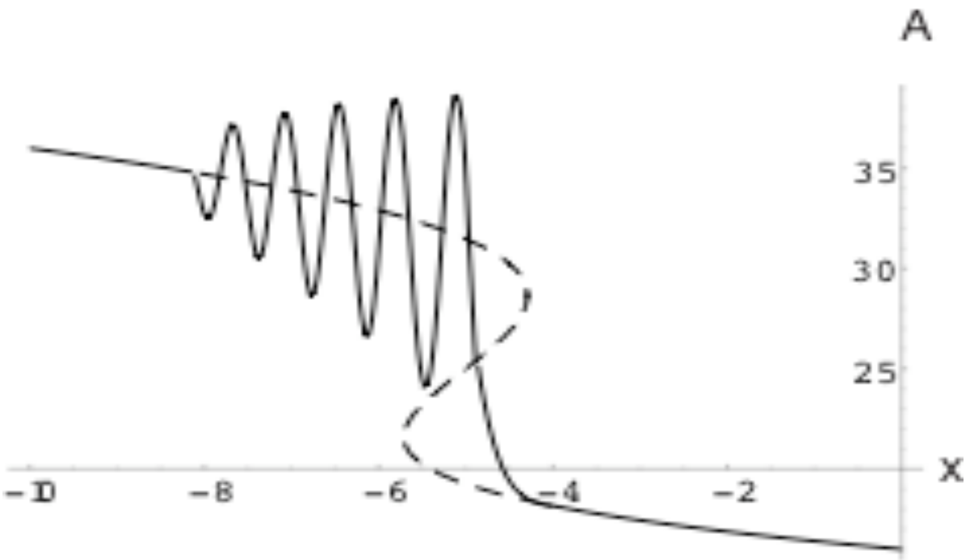


$\Delta = 0.2$ (F = 1.2) L = 50

Anagalous simulations of full Euler equations for surface waves: Δ = 0 (F=1) L=50 (Zhang & Chwang 2001)



Undular Bore : **This can be modelled as the outcome from a steepening wave-front in the framework of a nonlinear, dispersive wave equation, e.g. the KdV equation.**



Use Whitham modulation theory, Whitham (1974) and Gurevich & Pitaevskii (1974).

Korteweg-de Vries equation:

$$A_t + 6AA_x + A_{xxx} = 0$$

One-phase travelling periodic wave, cnoidal wave, with 3 free parameters, say modulus m, $0<m<1$, mean level d and wavenumber k.

$$A = a\{b(m)+\mathrm{cn}^2(k[x-ct];m)\}+d$$

where $b(m) = \{[1-m]K(m) - E(m)\}/\{mK(m)\}$

speed: $c = 6d + 2a\{[2-m]K(m) - 3E(m)\}/\{mK(m)\}$

amplitude: $a = 2mk^2$

$m \to 1$ is the KdV solitary wave, "sech^2"

$m \to 0$ are small-amplitude sinusoidal waves

Whitham modulation theory develops equations for the modulation of the parameters (amplitude, wavelength, speed, mean, modulus m of the cnoidal function) of an exact one-phase periodic travelling wave solution. These modulation equations are obtained by averaging conservation laws, and form (3) nonlinear hyperbolic equations, which are diagonalizable, and have a similarity solution $m = m(x/t)$ for the case when the initial condition is a step-discontinuity of height $H > 0$.

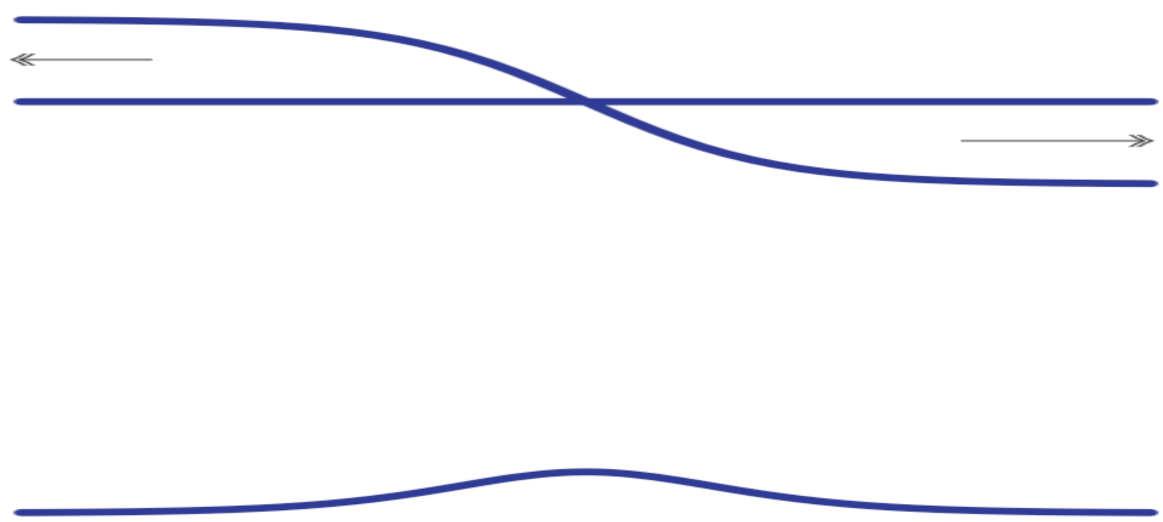
Leading edge, $m \to 1$: Soliton with amplitude 2H, twice that of the initial jump

Trailing edge, $m \to 0$: Small-amplitude sinusoidal waves.

Unsteady: $-6Ht < x < 4Ht$

Localized (Positive) Forcing

The key is the existence, in the transcritical regime, of a localized steady "hydraulic" solution in the forcing region. This is characterized by an upstream constant state A_- (>0) and a downstream constant state A_+ (<0). These states are resolved back to the zero state by undular bores, propagating upstream and downstream respectively.



Hydraulic approximation:

$$-A_t - \Delta A_x + 6AA_x + [A_{xxx}] + G_x(x) = 0$$

Steady state ($A_t = 0$) implies that: $-\Delta A + 3A^2 + G(x) = C$ (constant)

The constant is determined by the criticality condition at the peak of the forcing, $A_x \neq 0$ at $G_x = 0$, $G = G_m$ (consider the evolution of the characteristics). Since, for localized forcing, $G \rightarrow 0$ upstream and downstream, it follows that $A \rightarrow A_\pm$ upstream and downstream, where

$$6A_+ = \Delta - (12G_m)^{1/2} < 0 < 6A_- = \Delta + (12G_m)^{1/2} \ .$$

These expressions hold in the transcritical regime:

$$-(12G_m)^{1/2} < \Delta < (12G_m)^{1/2} \ .$$

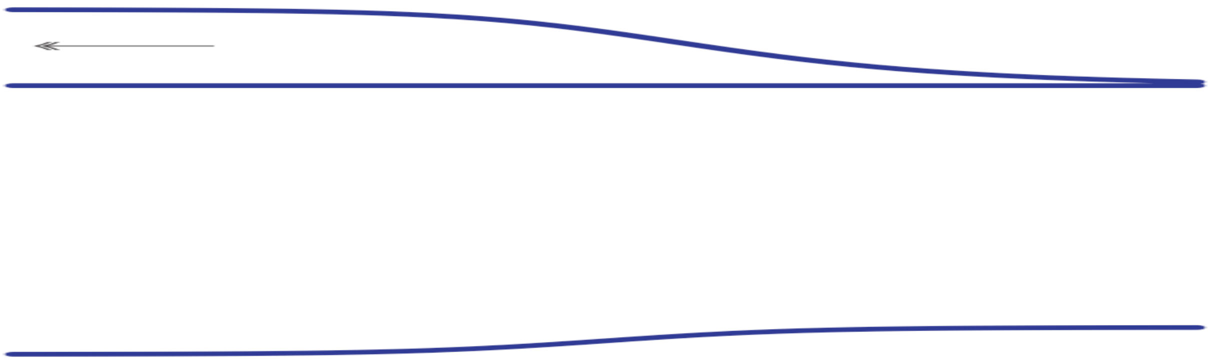
Since $A_- > 0 > A_+$ the corresponding jumps are resolved by undular bores propagating upstream and downstream respectively. Because the undular bores must lie in x<0 (>0) respectively, it follows that:

$-(12G_m)^{1/2} < \Delta < -(3G_m)^{1/2}$: upstream fully detached, downstream attached

$-(3G_m)^{1/2} < \Delta < (12G_m)^{1/2}$: upstream attached, downstream fully detached

Flow over a step

The same strategy is employed, namely we seek the locally steady hydraulic solutions, considering positive and negative steps separately. This is valid until the time when a disturbance from the step (down) at x=L reaches the step (up) at x=0. After that, the solution will readjust to the same state just described for localized forcing.



The essential difference that will emerge is that the downstream state A_+ at x=0 (upstream state A_- at x=L) will not produce an undular bore, and instead produces a rarefraction wave. However, the upstream state A_+ at x =0 (downstream state A_+ at x=L) will still produce an undular bore.

As before, use the hydraulic approximation:

$$-A_t - \Delta A_x + 6AA_x + [A_{xxx}] + G_x(x) = 0$$

Steady state ($A_t = 0$) implies that: $-\Delta A + 3A^2 + G(x) = C$ (constant)

But now, the constant C is not determined by a criticality condition; instead it is found by considering the evolution of the characteristics from the flat regions before and after the step.

First, consider the step up at x=0. Then we find that:

$\Delta < 0, \quad 6A_- = \Delta + (\Delta^2 + 12G_m)^{1/2}, \quad 6A_+ = 0,$

$0 < \Delta < (12G_m)^{1/2}, \quad 6A_- = \Delta + (12G_m)^{1/2}, \quad 6A_+ = \Delta,$

$(12G_m)^{1/2} < \Delta, \quad 6A_- = 0, \quad 6A_+ = \Delta - (\Delta^2 - 12G_m)^{1/2}.$

Now $A_- > 0$ for all $\Delta < (12G_m)^{1/2}$, and so leads to an upstream propagating undular bore. It is fully detached for $\Delta < -(4G_m)^{1/2}$, but is attached for $-(4G_m)^{1/2} < \Delta < (12G_m)^{1/2}$, and is zero for $(12G_m)^{1/2} < \Delta$.

On the other hand, $A_+ > 0$ for all $\Delta > 0$, and is then terminated by a rarefraction wave; no undular bore is needed. For $\Delta < 0$, A_+ is zero.

Next, consider the step down at x=L. Instead we find that:

$\Delta > 0, \quad 6A_+ = \Delta - (\Delta^2 + 12G_m)^{1/2} \ , \ 6A_- = 0 \ ,$

$0 > \Delta > -(12G_m)^{1/2}, \quad 6A_+ = \Delta - (12G_m)^{1/2} \ , \ 6A_- = \Delta \ ,$

$-(12G_m)^{1/2} > \Delta \ , 6A_+ = 0, \quad 6A_- = \Delta + (\Delta^2 + 12G_m)^{1/2} \ .$

Now $A_+ < 0$ for all $\Delta > -(12G_m)^{1/2}$, and so leads to an downstream propagating undular bore. It is fully detached for $\Delta > -(3G_m)^{1/2}$, but is attached for $-(12G_m)^{1/2} < \Delta < -(3G_m)^{1/2}$, and is zero for $-(12G_m)^{1/2} > \Delta$.

On the other hand, $A_- < 0$ for all $\Delta < 0$, and is then terminated by a rarefraction wave; no undular bore is needed. For $\Delta > 0$, A_+ is zero.

The properties of the respective undular bores in both cases are completely determined by the respective values of A_+ and A_-

Results table

Δ	fKdV				Theory			
	A_{wu}	A_u	A_{wd}	A_d	A_{wu}	A_u	A_{wd}	A_d
0.2	0.83	0.44	0.31	-0.16	0.80	0.40	0.32	-0.16
0.1	0.66	0.45	0.39	-0.20	0.66	0.33	0.40	-0.20
0.0	0.50	0.30	0.51	-0.26	0.52	0.26	0.52	-0.26
-0.1	0.39	0.22	0.64	-0.33	0.40	0.20	0.66	-0.33
-0.2	0.30	0.16	0.84	-0.40	0.32	0.16	0.80	-0.40
-0.3	0.24	0.13	0.64	-0.38	0.26	0.13	0.92	-0.46
-0.4	0.19	0.10	0.00	0.00	0.20	0.10	0.00	0.00

Note: $A_u = A_-$ upstream , $A_d = A_+$ downstream ,
A_{wu} = wave amplitude upstream,
A_{wd} = wave amplitude downstream

Δ	fKdV				Euler			
	A_{wu}	A_u	A_{wd}	A_d	A_{wu}	A_u	A_{wd}	A_d
0.2	0.83	0.44	0.31	-0.16	0.75	0.61	0.28	-0.18
0.1	0.66	0.45	0.39	-0.20	0.57	0.31	0.32	-0.21
0.0	0.50	0.30	0.51	-0.26	0.44	0.33	0.37	-0.25
-0.1	0.39	0.22	0.64	-0.33	0.32	0.20	0.43	-0.30
-0.2	0.30	0.16	0.84	-0.40	0.23	0.13	0.53	-0.36
-0.3	0.24	0.13	0.64	-0.38	0.16	0.08	0.57	-0.38
-0.4	0.19	0.10	0.00	0.00	0.10	0.01	0.00	0.00

References:

R.H.J. Grimshaw and N.F. Smyth, J. Fluid Mech., 169: 429-464, 1986.

D-H. Zhang and A.T. Chwang, J. Fluid Mech., 432: 341-350, 2001.

R.H.J. Grimshaw, D-H. Zhang and K.W. Chow, J. Fluid Mech., 2007 (to appear)

The end

Thank you