Effects of dissipative disorder on front formation in pattern forming systems

Avner Peleg[1,2], Tomáš Dohnal[3,2] and Yeojin Chung[2,3]
[1] *Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA*
[2] *Theoretical Division, LANL, Los Alamos, NM 87545, USA and*
[3] *Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA*

We study the effects of weak disorder in the linear gain coefficient on front formation in pattern forming systems described by the cubic-quintic nonlinear Schrödinger equation. We calculate the statistics of the front amplitude and position. We show that the distribution of the front amplitude has a loglognormal diverging form at the maximum possible amplitude and that the distribution of the front position has a lognormal tail. The theory is in good agreement with our numerical simulations. We show that these results are valid for other types of dissipative disorder and relate the loglognormal divergence of the amplitude distribution to the form of the emerging front tail.

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The theory of pattern formation aims to explain the generation and dynamic evolution of the multitude of patterns appearing in nature. This theory is of major importance in a wide variety of fields ranging from physics and chemistry to biology and social sciences. Most pattern forming systems in these fields are affected by processes involving noise or disorder. When the disorder is strong, the patterns are usually destroyed. However, when the disorder is weak, the patterns can form and evolve under the influence of the disorder and a very important question arises regarding the statistics of the parameters characterizing the forming patterns.

Fronts are probably among the most common patterns appearing in nature. In this paper we investigate front formation induced by dissipative disorder, i.e., disorder in the linear or nonlinear gain coefficient, in systems described by the cubic-quintic nonlinear Schrödinger equation (CQNLSE). The CQNLSE is one of the simplest nonintegrable extensions to the cubic nonlinear Schrödinger equation (CNLSE) for which solitary wave solutions are readily obtained [1–7]. The CQNLSE and its extension, the cubic-quintic complex Ginzburg-Landau equation (CQCGLE), are of special interest, since they describe a wide variety of nonlinear physical phenomena. Examples include pulse propagation in certain optical fibers and waveguides [6, 8], convection in pure and binary fluids [7], mode-locked lasers [9] and plasma-laser interaction [10]. Moreover, the CQCGLE, which is a generalization of the cubic complex Ginzburg-Landau equation (CCGLE), is expected to be valid in systems described by the CCGLE whenever the cubic coefficient is small [1, 2]. Because of the nonintegrability of the CQNLSE the dynamics of its solitary waves is far richer than the dynamics of solitons of the CNLSE. In particular, small perturbations can lead to front formation, pulse splitting, oscillation or collapse. Due to this richness, the CQNLSE has been widely used as a basic model in pattern formation [1, 2].

Solitary waves, fronts and other travelling wave solutions for the CQNLSE and for the CQCGLE were obtained in several earlier works [1–7]. Much attention was devoted to analyzing the stability of these solutions under small perturbations [4, 5, 7, 11]. These studies have shown that the solitary waves either decay or evolve into fronts in the presence of a small linear loss or gain term, respectively. Furthermore, it was noted that in the case of linear gain the solitary waves are unstable with respect to emission of continuous radiation. Effects of noise and disorder on solitons of the CNLSE [12] and on patterns appearing in the CCGLE [13] have been studied in detail. However, to the best of our knowledge, effects of dissipative disorder on the evolution of stationary patterns of the CQNLSE and the CQCGLE have never been investigated before.

We focus our attention on disorder in the linear gain coefficient since such disorder is expected to be quite common in a variety of physical systems. In fiber lasers and in optical fiber transmission, for example, such disorder exists when the linear gain/loss coefficient is fluctuating with propagation distance. In addition, it was recently shown that effective disorder in the linear gain coefficient plays an important role in optical fiber telecommunication systems with multiple frequency channels [14, 15]. In pattern forming systems where the patterns are spatial such disorder can exist as a result of fluctuations with respect to time in the energy that is pumped into or out of the system. Indeed, a deterministic linear gain/loss term is included as a part of the CCGLE and CQCGLE and plays a special role in the dynamics [1, 2]. Furthermore, the solitary waves of the CQNLSE become unstable in the presence of linear gain and evolve into fronts. It is therefore very interesting to study the situation where the linear gain/loss coefficient is stochastic. The most important questions in this case are: (1) What are the statistics of the emerging fronts parameters? (2) Are these statistics dependent or independent of the form of the dissipative disorder term?

In this paper we address these questions. Using the adiabatic perturbation theory, we obtain the dynamics of the solitary wave amplitude. We calculate the amplitude distribution and show that it has a loglognormal diverging form in the vicinity of the maximum possible amplitude. We then calculate the distribution of the front position and find that it has a lognormal tail. These predictions are in good agreement with our extensive numerical simulations. We show that the loglognormal and lognormal asymptotic behavior of the distribution functions are valid for other types of dissipative disorder and relate the loglognormal divergence of the amplitude distribution to the shape of the tail of the emerging front.

Consider the CQNLSE in the presence of weak disorder in the linear gain coefficient, where the disorder is spatially short correlated and zero in average:

$$\begin{aligned} &i\psi_z+\psi_{tt}+2|\psi|^2\psi-\varepsilon_q|\psi|^4\psi=i\epsilon\xi(z)\psi, \\ &\langle\xi(z)\rangle=0, \quad \langle\xi(z)\xi(z')\rangle=D\delta(z-z'). \end{aligned} \quad (1)$$

In the context of nonlinear optics ψ is the envelope of the electric field, z is the propagation distance, t is a retarded time, ε_q is the quintic coefficient, $0<\epsilon\ll 1$ is the linear gain coefficient and D is the disorder intensity. The terms $\varepsilon_q|\psi|^4\psi$ and $i\epsilon\xi(z)\psi$ account for quintic nonlinearity and disorder in the linear gain coefficient, respectively. When $\epsilon=0$, Eq. (1) supports solitary wave solutions of the form $\psi_s(t,z)=\Psi_s(x)\exp(i\chi)$, where

$$\Psi_s(x)=\sqrt{2}\eta\left[(1-4\varepsilon_q\eta^2/3)^{1/2}\cosh(2x)+1\right]^{-1/2}, \quad (2)$$

$\chi=\alpha+\beta(t-y)+(\eta^2-\beta^2)z$ and $x=\eta(t-y-2\beta z)$. In these relations the parameters η,β,α,y are related to the amplitude, frequency, phase and position of the solitary wave, respectively. Note that the solitary wave solution ψ_s exists provided that $\eta<\eta_m\equiv(4\varepsilon_q/3)^{-1/2}$.

We study the evolution of the solitary wave ψ_s under the dynamics described by Eq. (1). Since we are interested in the statistics of front formation, we restrict the discussion to the case $\varepsilon_q>0$. We also assume that $4D\epsilon^2z\ll 1$, so that for most of the disorder realizations the dynamics of the solitary wave amplitude is not yet influenced by the $O(\epsilon^2)$ radiation instability effects [1, 5, 11]. Due to the symmetry of Eq. (1) with respect to $t\to -t$ the position and frequency of the solitary waves are not affected by the disorder. The dynamics of the amplitude can be obtained by using the energy conservation law:

$$\partial_z\int_{-\infty}^{\infty}\mathrm{d}t|\psi|^2=2\epsilon\xi(z)\int_{-\infty}^{\infty}\mathrm{d}t|\psi|^2. \quad (3)$$

Employing the adiabatic perturbation theory around the solitary waves ψ_s (see, e.g., Ref. [1]) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\left[\operatorname{arctanh}\left(\frac{\eta}{\eta_m}\right)\right]=2\epsilon\xi(z). \quad (4)$$

Integrating Eq. (4) over z we arrive at

$$\eta(z)=\eta_m\tanh\left\{c(0)\exp\left[2\epsilon x(z)\right]\right\}, \quad (5)$$

where $x(z)=\int_0^z \mathrm{d}z'\xi(z')$ and $c(0)=\operatorname{arctanh}[\eta(0)/\eta_m]$. Since $\xi(z)$ is short correlated, according to the central limit theorem $x(z)$ is a Gaussian random variable with $\langle x(z)\rangle=0$ and $\langle x^2(z)\rangle=Dz$. The distribution function of η is obtained by changing variables from $x(z)$ to $\eta(z)$ while using Eq. (5). This yields

$$F(\eta)=\frac{\exp\left\{-\ln^2[\operatorname{arctanh}(\eta/\eta_m)/c(0)]/(8D\epsilon^2z)\right\}}{(8\pi D\epsilon^2z)^{1/2}\eta_m\left(1-\eta^2/\eta_m^2\right)\operatorname{arctanh}(\eta/\eta_m)} \quad (6)$$

for $0\le\eta<\eta_m$ and $F(\eta)=0$ elsewhere. For $\varepsilon_q\ll 1$ the distribution (6) approaches the lognormal distribution, which was obtained for propagation of CNLSE solitons in the presence of weak disorder in the linear gain coefficient [14, 15]. Notice, however, that for finite values of ε_q $F(\eta)$ is very different from the lognormal distribution. First, $F(\eta)$ has a compact support. Second, $F(\eta)\to\infty$ as $\eta\to\eta_m$. This can be shown by considering the asymptotic behavior of $F(\eta)$ near η_m. Denoting $\eta=\eta_m-\delta\eta$, where $0<\delta\eta/\eta_m\ll 1$, and expanding (6) with respect to $\delta\eta/\eta_m$ we obtain

$$\begin{aligned} F(\delta\eta)|_{\eta\lesssim\eta_m}=&\left\{(8\pi D\epsilon^2z)^{1/2}\delta\eta\left|\ln\left[\delta\eta/(2\eta_m)\right]\right|\right\}^{-1} \\ &\times\exp\left\{-\ln^2[-\ln\left[\delta\eta/(2\eta_m)\right]/(2c(0))]/(8D\epsilon^2z)\right\}. \end{aligned} \quad (7)$$

We refer to the distribution (7) as the *loglognormal* distribution. Third, when the disorder strength $4D\epsilon^2z$ is smaller than some threshold $\mathcal{D}$: $4D\epsilon^2z<\mathcal{D}(\varepsilon_q,\eta(0))$, $F(\eta)$ has a local minimum in the vicinity of η_m and a local maximum at some intermediate η. Using Eq. (7) we find that the location of the local minimum is approximately given by the equation $\ln[X/(2c(0))]=4D\epsilon^2z(X-1)$, where $X\equiv-\ln[\delta\eta/(2\eta_m)]$. When $4D\epsilon^2z>\mathcal{D}(\varepsilon_q,\eta(0))$, this equation does not have a solution and $F(\eta)$ does not have a local minimum (or a local maximum).

The dynamics described by Eq. (1) can either lead to front formation or to decay of the solitary waves. To analyze the dynamics of the emerging fronts we define the front position t_{fr} as the value of t for which $|\psi|=|\psi|_{max}/2$, where $|\psi|_{max}$ is the maximum of $|\psi|$ at a given z. Using Eq. (2) we obtain

$$t_{fr}=\operatorname{arccosh}\left[4+3(1-\eta^2/\eta_m^2)^{-1/2}\right]/(2\eta). \quad (8)$$

We say that the field $\psi(t,z)$ at a given z corresponds to a front if $B\eta_m\le\eta(z)<\eta_m$, where $B=0.95$. Since B is sufficiently close to 1, $t_{fr}(\eta)$ is a monotonously increasing function. Therefore, the distribution of the front position $G(t_{fr})$ is given by

$$G(t_{fr})=C\left(\frac{\mathrm{d}t_{fr}}{\mathrm{d}\eta}\right)^{-1}F(\eta(t_{fr})), \quad (9)$$

where $C=\left[\int_{B\eta_m}^{\eta_m}\mathrm{d}\eta F(\eta)\right]^{-1}$. Equations (8-9) uniquely define the distribution $G(t_{fr})$. The tail of this distribution is lognormal. To see this consider the dynamics given by Eq. (3) in the limit $\eta \to \eta_m$. In this limit the integral over t of $|\psi|^2$ can be approximated by $b\eta_m t_{fr}$, where b is a constant. Hence, t_{fr} satisfies $\mathrm{d}t_{fr}/\mathrm{d}z \simeq 2\epsilon\xi(z)t_{fr}$. Integrating over z we obtain $t_{fr} \simeq \text{const} \times \exp[2\epsilon x(z)]$, from which it follows that t_{fr} is lognormally distributed. A more rigorous calculation using Eqs. (7) and (8) yields

$$G(t_{fr})|_{t_{fr}\gg 1} \simeq \left\{(2\pi D\epsilon^2 z)^{1/2}(2\eta_m t_{fr} - \ln 3)\right\}^{-1} \times C\eta_m \exp\left\{\frac{-\ln^2\left[(2\eta_m t_{fr} - \ln 3)/c(0)\right]}{8D\epsilon^2 z}\right\}. \quad (10)$$

To check our theoretical predictions we performed numerical simulations with Eq. (1). We used an initial condition in the form of the solitary wave ψ_s with $\eta(0) = 1$, $\beta(0) = 0$, $y(0) = 0$, and $\alpha(0) = 0$. The following two sets of parameters were considered: $D = 3$, $\varepsilon_q = 0.5$, $\epsilon = 0.03$ and $\epsilon = 0.05$. The simulations were carried out up to a distance $z_f = 10$. For this value of z the disorder strength $4D\epsilon^2 z$ is 0.108 for $\epsilon = 0.03$ and 0.3 for $\epsilon = 0.05$.

Equation (1) was integrated by employing a split-step method that is of sixth order with respect to the z-step dz [16]. To overcome numerical errors resulting from radiation emission and the use of periodic boundary conditions we applied artificial damping in the vicinity of the boundaries of the computational domain. The size of the domain was taken to be $-L \le t \le L$ with $L = 10\pi$ so that the absorbing layers do not affect the dynamics of the solitary waves. For both values of ϵ we sampled more than 2.5×10^5 disorder realizations. In the $\epsilon = 0.03$ case we employed the technique of importance sampling [17] in order to access the tail of $G(t_{fr})$.

The amplitude distributions $F(\eta)$ at $z_f = 10$ obtained by our numerical simulations for $\epsilon = 0.03$ and $\epsilon = 0.05$ are shown in Figs. 1(a) and 1(b), respectively. A comparison with the theoretical prediction given by Eq. (6) is also presented. The insets show a blow up of the same data in the vicinity of η_m together with the theoretical prediction for the asymptotic form given by Eq. (7). One can see that the behavior of $F(\eta)$ is quite different in the two cases. While for $\epsilon = 0.03$ $F(\eta)$ possesses a local maximum at an intermediate value $\eta_0 \simeq 1.06$, for $\epsilon = 0.05$ it does not have a local maximum or minimum. Instead, in the latter case $F(\eta)$ is quite flat at $1 < \eta < 1.2$ and then increases sharply at $1.2 < \eta < \eta_m$. This is a result of the fact that for $\epsilon = 0.05$ $4D\epsilon^2 z_f > \mathcal{D}(\epsilon_q = 0.5, \eta(0) = 1)$, i.e., the final disorder strength is intermediate. In both cases the agreement between theory and simulations is good in the main body of the distribution as well as in the vicinity of η_m. The later statement means that the asymptotic behavior of $F(\eta)$ near η_m is indeed loglognormal in both cases. In the $\epsilon = 0.03$ case, $F(\eta)$ has a minimum located within $\Delta\eta \simeq 10^{-5}$ from η_m and then increases sharply with increasing η, but we were unable to capture this very narrow region with our simulations. We were able to capture the minimum of $F(\eta)$ in the $\epsilon = 0.05$ case at $z = 6$. The corresponding $F(\eta)$ curve is shown in Fig. 2.

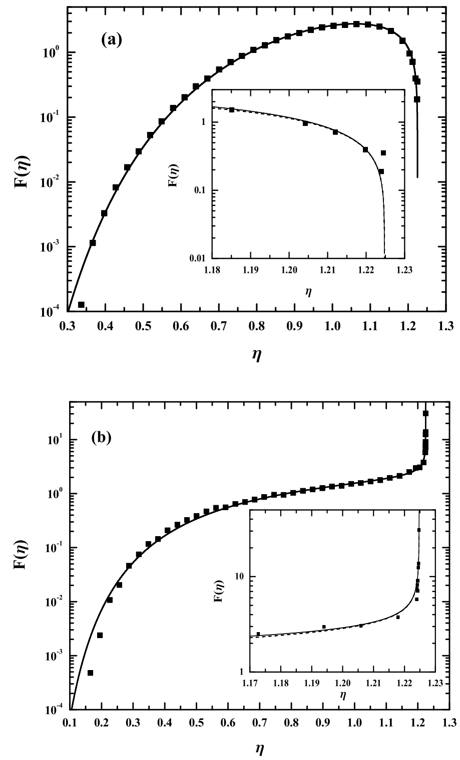

FIG. 1: Distribution function of the pulse amplitude $F(\eta)$ at $z_f = 10$ for $D = 3$, $\varepsilon_q = 0.5$, $\epsilon = 0.03$ (a) and $\epsilon = 0.05$ (b). The squares represent the result obtained by numerical simulations, while the solid lines correspond to the theoretical prediction given by Eq. (6). The insets show a blow up of the same data in the vicinity of η_m. The dashed lines stand for the asymptotic loglognormal form given by Eq. (7).

Figures 3(a) and 3(b) show the distribution functions of the front position $G(t_{fr})$ at $z_f = 10$ as obtained by the numerical simulations with $\epsilon = 0.03$ and $\epsilon = 0.05$, respectively. The figures also present the theoretical prediction given by Eqs. (8-9) together with the asymptotic lognormal form (10). The agreement between theory and simulations is very good for $\epsilon = 0.03$. In the $\epsilon = 0.05$ case good agreement is obtained for $t_{fr} < 5$, i.e., in the near part of the tail, while for $t_{fr} > 5$ there are not enough realizations to make a clear statement. From this we conclude that the near tail of the distribution of the front position is indeed lognormal.

We now show that the loglognormal and lognormal character of the tails of $F(\eta)$ and $G(t_{fr})$, respectively, are valid for other types of dissipative disorder. Consider the dynamics of the solitary wave ψ_s in the presence of a dissipative disorder $i\epsilon\xi(z)|\psi|^{2n}\psi$, where $n \ge 0$ is an integer. The cases $n = 1, 2$ are of special importance since corresponding deterministic terms are incorporated in the CCGLE and the CQCGLE [1, 2]. We assume that

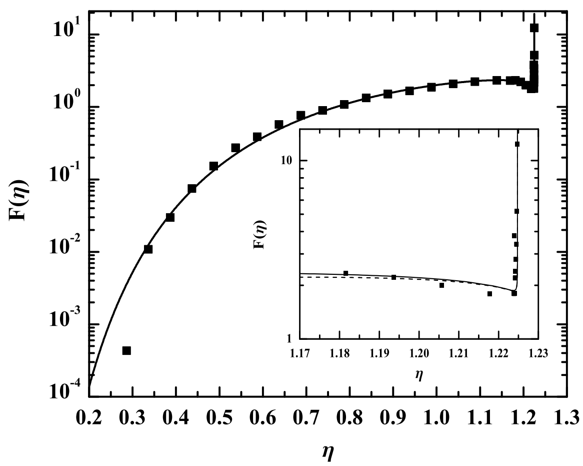

FIG. 2: Distribution function of the pulse amplitude $F(\eta)$ at $z = 6$ for $D = 3$, $\varepsilon_q = 0.5$ and $\epsilon = 0.05$. The squares correspond to the result obtained by numerical simulations, while the solid line represents the theoretical prediction given by Eq. (6). The inset shows a blow up of the same data in the vicinity of η_m. The dashed line stand for the asymptotic loglognormal form given by Eq. (7).

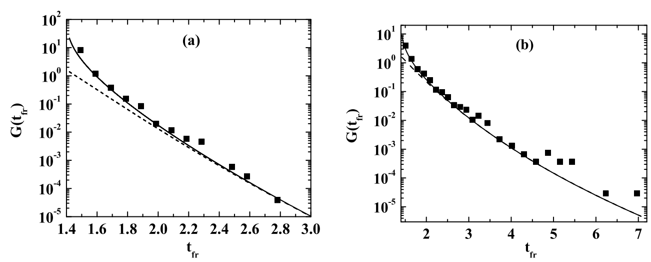

FIG. 3: Distribution function of the front position $G(t_{fr})$ at $z_f = 10$ for $D = 3$, $\varepsilon_q = 0.5$, $\epsilon = 0.03$ (a) and $\epsilon = 0.05$ (b). The squares represent the result obtained by numerical simulations. The solid and dashed lines correspond to the theoretical prediction [Eqs. (8-9)] and to the asymptotic lognormal form [Eq. (10)], respectively.

$\varepsilon_q^{1/2}$ is not small, so that η_m^{2n} is not a large parameter. Using energy conservation we obtain an equation similar to Eq. (3) for the amplitude dynamics. Taking the limit $\eta \to \eta_m$ in that equation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \ln \left[\frac{e^{2c_n} \delta\eta}{2\eta_m} \right] \right\} = 2^{n+1} \eta_m^{2n} \epsilon \xi(z), \qquad (11)$$

where c_n is a constant. Integrating and solving for $x(z)$ we find that $F(\eta)$ is loglognormal in the vicinity of η_m

$$F(\delta\eta)|_{\eta \lesssim \eta_m} \simeq \left\{ (32\pi D\epsilon^2 z)^{1/2} 2^n \eta_m^{2n} \delta\eta \left| \ln \left[\frac{e^{2c_n} \delta\eta}{2\eta_m} \right] \right| \right\}^{-1}$$
$$\times \exp \left\{ - \frac{\ln^2 \left[- \ln \left[(e^{2c_n} \delta\eta)/(2\eta_m) \right] / (2\tilde{c}) \right]}{2^{2n+3} \eta_m^{4n} D\epsilon^2 z} \right\}, \qquad (12)$$

where $\tilde{c}$ is another constant. In a similar manner we obtain that the tail of $G(t_{fr})$ is lognormal

$$G(t_{fr})|_{t_{fr} \gg 1} \simeq \left\{ (2\pi D\epsilon^2 z)^{1/2} 2^n \eta_m^{2n} (2\eta_m t_{fr} - \ln 3 - c_n) \right\}^{-1}$$
$$\times C\eta_m \exp \left\{ \frac{-\ln^2 \left[(2\eta_m t_{fr} - \ln 3 - c_n)/\tilde{c} \right]}{2^{2n+3} \eta_m^{4n} D\epsilon^2 z} \right\}. \qquad (13)$$

The result (12) means that the amplitude statistics for solitary waves of the CQNLSE are very different from the statistics for CNLSE solitons. Indeed, in the latter case, where $\varepsilon_q = 0$, the disorder $i\epsilon\xi(z)|\psi|^{2n}\psi$ can lead to an exponential growth of the amplitude associated with the lognormal tail of $F(\eta)$ for $n = 0$ and to a blow up of the amplitude after a finite propagation distance for $n \geq 1$. Near the integrable limit, i.e., when $\varepsilon_q^{1/2} \ll 1$ and $\eta_m \gg 1$, one can expect the amplitude statistics to be similar to the CNLSE case. For such values of ε_q the first order perturbation description might break down before the solitary waves approach the asymptotic front form and, therefore, we cannot make a statement about the behavior of $F(\eta)$ near η_m.

A simple argument shows that the loglognormal divergence of $F(\eta)$ is related to the asymptotic form of the emerging front tail. Let us concentrate on disorder in the linear gain. We already showed that in this case when $\eta \to \eta_m$ t_{fr} is lognormally distributed, i.e., $t_{fr} \simeq \mathrm{const} \times \exp[2\epsilon x(z)]$. Furthermore, taking the limit $t \gg 1$ and $\delta\eta/\eta_m \ll 1$ in Eq. (2) we obtain that the tail of the front is given by

$$\Psi_s(x)|_{t \gg 1, \eta \lesssim \eta_{max}} \sim (\delta\eta/\eta_m)^{-1/4} \exp(-\eta_m t). \qquad (14)$$

Using this asymptotic form at $t = t_{fr} \gg 1$ together with the definition of t_{fr} we obtain $t_{fr} \sim -\ln[\delta\eta/(2\eta_m)]/(4\eta_m)$, from which it follows that $F(\eta)$ is loglognormal in the vicinity of η_m. Thus, the loglognormal divergence of the amplitude distribution is indeed determined by the form of the emerging front tail. It would be interesting to see if similar statistical behavior exist in other types of pattern forming systems supporting front formation in the presence of dissipative disorder.

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