Eidgenössische Forschungsanastalt für Wald, Schnee und Landschaft

Abteilung Schnee und Lawninen

Collapse phenomenon in the NLS and Townes soliton

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Overview

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NLS with an attracting nonlinearity

The NLS models pulse propagation in an optical fiber at the lowest order of nonlinearity.

$$i\partial_t \Psi + \Delta\Psi + |\Psi|^{2\sigma}\Psi = 0$$
$$\Psi(\mathbf{x}, 0) = \phi(\mathbf{x})$$

Under special conditions, the pulse can self focus, which corresponds to a blowup of the solution.

- critical case: $\sigma d = 2$
- supercritical case: $\sigma d > 2$

(d: dimension of the problem)

Contents

Bound states and Ground states

A special class of solutions is derived by making the Ansatz:

$$\Psi(\mathbf{x}, t) = e^{i\lambda^2 t}\Phi(\mathbf{x})$$

They are referred to as: Standing waves, solitary waves, wave guides or bound states. Φ satisfies

$$\Delta\Phi + \lambda^2\Phi + |\Phi|^{2\sigma}\Phi = 0$$

Solutions:

- $d = 1$: unique solution
- $d > 1$: existence of a unique radially symmetric and positive solution (ground state)

The Towns profile is the ground state standing wave with $d = 2$ and $\sigma = 1$

Existence of the Townes profile 1

Task: Find solutions $g \in H^1(R^d)$ of

$$\Delta g - \lambda^2 g + g^{2\sigma+1} = 0 \tag{1}$$

Necessary conditions for the existence are the following (Pohozaev) identities:

$$\int |\nabla g|^2 \, d\mathbf{x} = \frac{\sigma d}{2(\sigma+1)} \int |g|^{2\sigma+2} \, d\mathbf{x}$$

$$\frac{\lambda^2}{d} \int |g|^2 \, d\mathbf{x} = \left(\frac{1}{d} - \frac{\sigma}{2(\sigma+1)}\right) \int |g|^{2\sigma+2} \, d\mathbf{x}$$

$\Rightarrow$ No solutions ($\in H^1(R^d)$), if $\sigma > \frac{2}{d-2}$

Existence of the Townes profile 2

Theorem

Suppose $d \geq 2$ and $\sigma < \frac{2}{d-2}$. (no extra condition on σ, if $d = 2$). Then (1) has a positive, spherically symmetric solution $g \in C^2(R^d)$. In addition, g and its derivatives up to second order have an exponential decay at infinity. This solution minimizes the Action, among all $H^1(R^d)$- solutions of (1).

Theorem

For $0 < \sigma < \frac{2}{d-2}$ the positive solution is unique.

Existence of the Townes profile 3

Idea of the proof:

- The Action is defined by $S(u) := \frac{1}{2}\,(\,H(u) + \lambda^2 N(u)\,)$

$$H(u) := \int (|\nabla u|^2 - \frac{1}{\sigma+1}\,|u|^{2\sigma+2})\ d\mathbf{x} \quad N(u) := \int |u|^2\ d\mathbf{x}$$

- S is a C^1-functional on $H^1(R^d)$
- For solutions g of (1) $S(g) > 0$
- Find $g := \min\{S(u) : u \in H^1(R^d), u \text{ is solution of } (1)\ \}$
- Show that g is positive
- Apply some theorem to get: g is spherically symmetric

Computation of the Towns Profile 1

We know that the solution of interest is radial symmetric, thus equation (1) transforms to

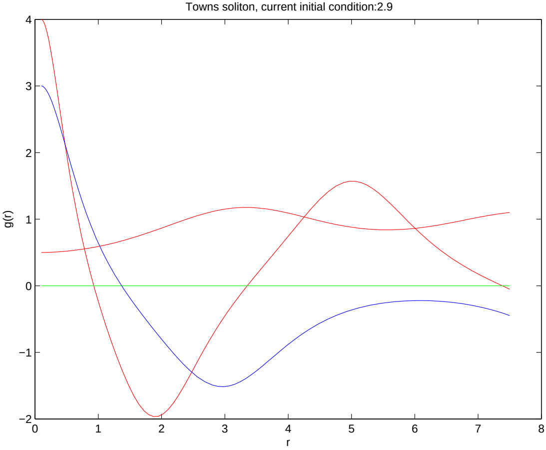
$$\frac{\partial^2}{\partial r^2} g + \frac{1}{r^2} \underbrace{\frac{\partial^2}{\partial \theta^2} g}_{=0} + \frac{1}{r} \frac{\partial}{\partial r} g + \lambda^2 g + g^{2\sigma+1} =$$

$$\frac{\partial^2}{\partial r^2} g + \frac{1}{r} \frac{\partial}{\partial r} g + \lambda^2 g + g^{2\sigma+1} = 0$$

This second order ODE has to be solved with respect to the following boundary conditions:

- $\frac{\partial}{\partial r} g|_{r=0} \; = \; 0$
- $\lim_{r\to\infty} \; g(r) \; = \; 0$

Computation of the Towns Profile with shooting



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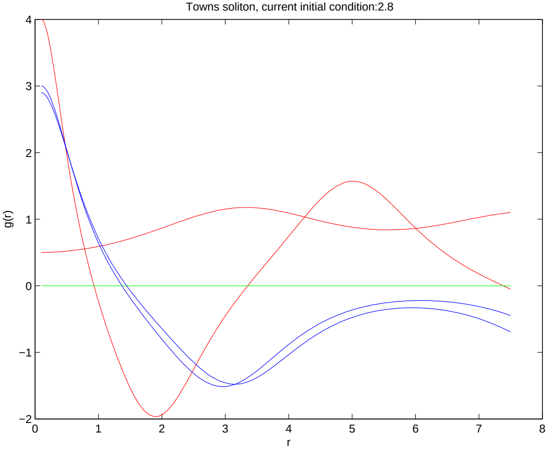

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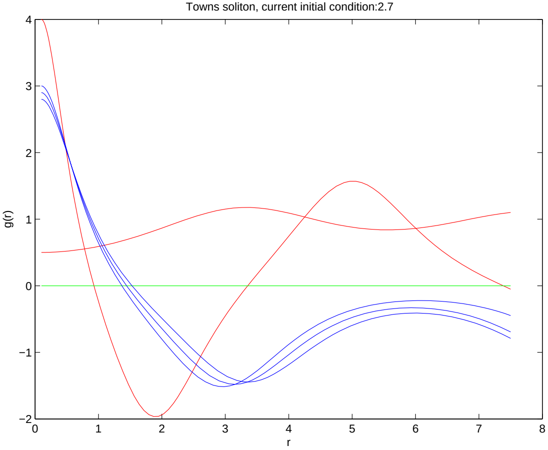

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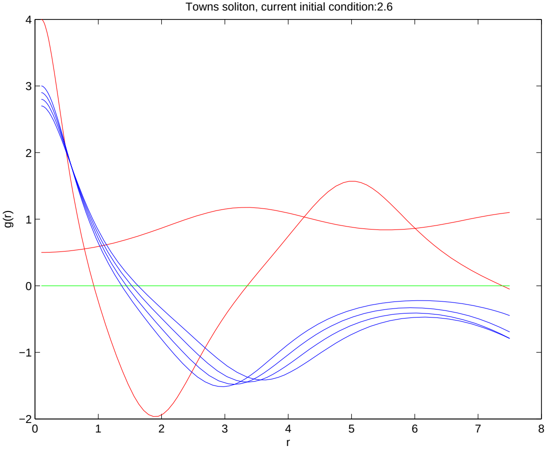

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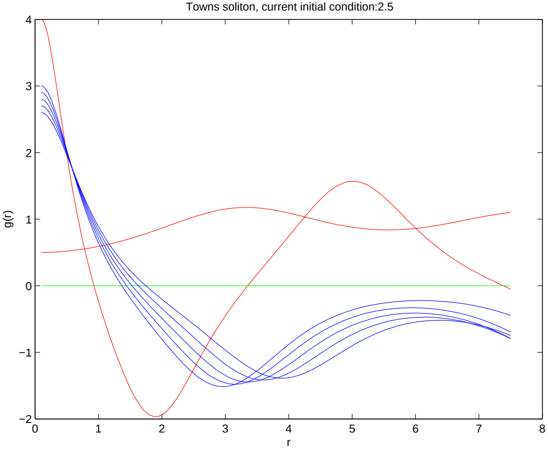

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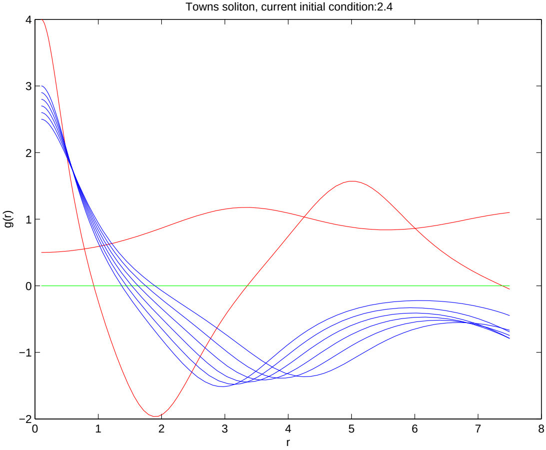

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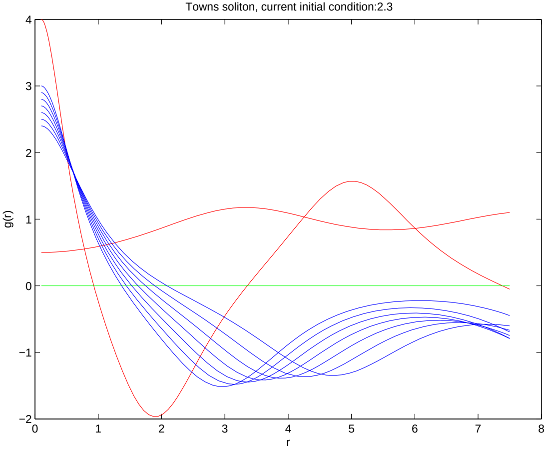

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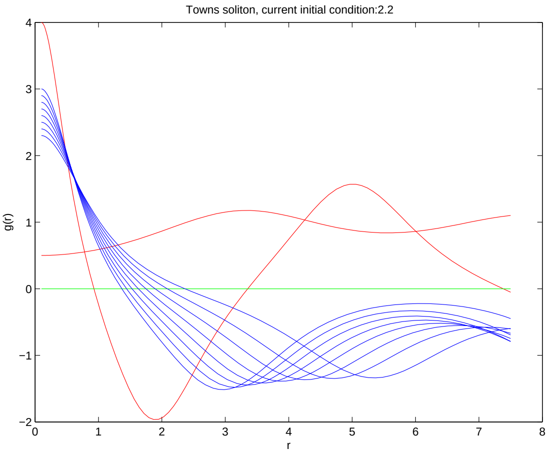

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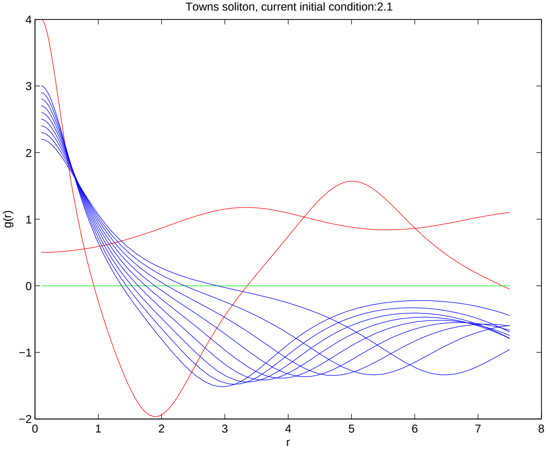

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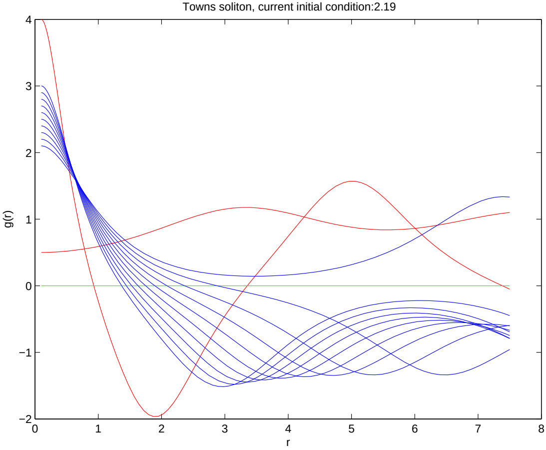

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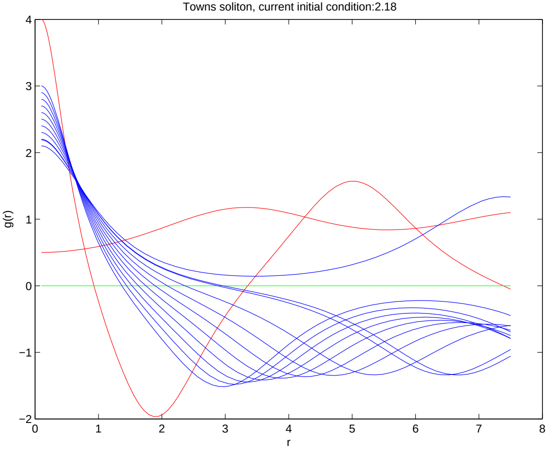

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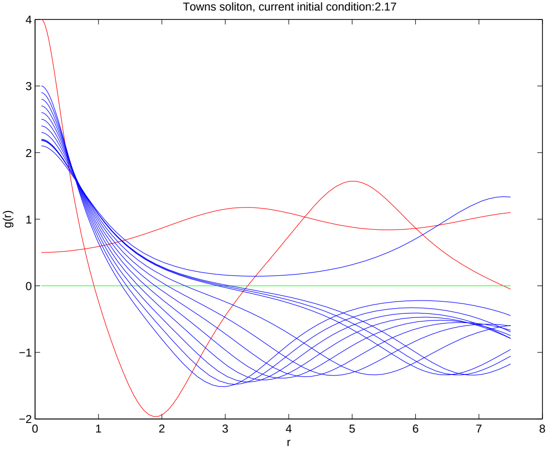

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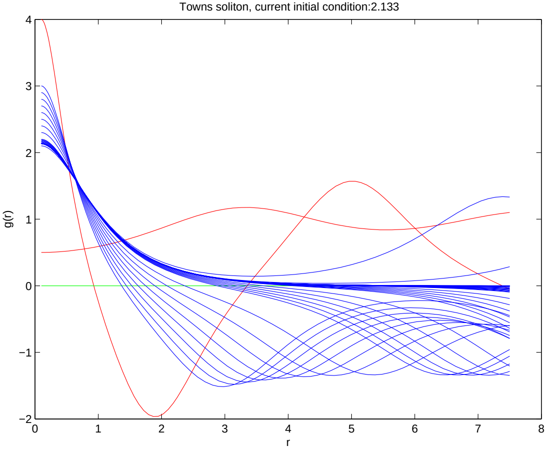
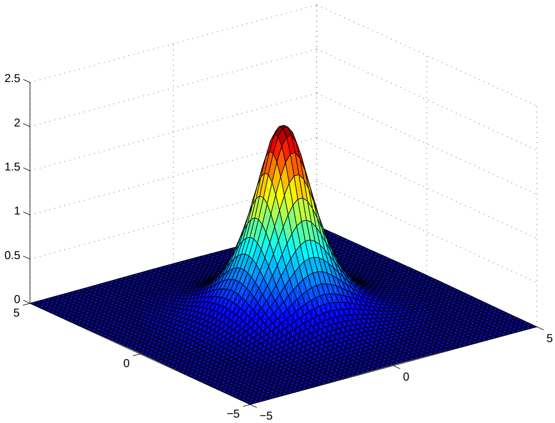


Towns Profile



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Finite Time Blowup

Theorem

Suppose that $\sigma d \geq 2$. *Consider an initial condition* $\phi \in H^1$ *with* $V(0) < \infty$, *that satisfies one of the conditions below:*

- $H(\phi) < 0$
- $H(\phi) = 0$ *and* $V'(0) < 0$
- $H(\phi) > 0$ *and* $V'(0) \leq -4\sqrt{H(\phi)}|\mathbf{x}\phi|_{L^2}$

Then, there exists a time $t_* < \infty$ *such that*

$$\lim_{t \to t_*} |\nabla \Psi|_{L^2} = \infty \quad \textit{and} \quad \lim_{t \to t_*} |\Psi|_{L^\infty} = \infty$$

Useful stuff

- Variance

$$V(t) := \int |\mathbf{x}|^2 |\Psi|^2 \, d\mathbf{x}$$

- Variance Identity

$$\frac{1}{8}\frac{d^2}{dt^2} V(t) = H - \frac{d\sigma - 2}{2\sigma + 2} \int |\Psi|^{2\sigma+2} \, d\mathbf{x}$$

- Some L^2–estimation

$$\int |f|^2 \, d\mathbf{x} = \frac{1}{d} \int (\nabla \cdot \mathbf{x}) |f|^2 \, d\mathbf{x} =$$

$$= -\frac{1}{d} \int \mathbf{x} \cdot \nabla |f|^2 \, d\mathbf{x} = -\frac{1}{d} \int \mathbf{x} \cdot 2|f| \nabla |f| \, d\mathbf{x}$$

$$\Rightarrow |f|_{L^2}^2 \leq \frac{2}{d} |\nabla f|_{l^2} |\mathbf{x} f|_{L^2}$$

Proof 1

Suppose $d\sigma = 2$, then there holds

$$\frac{d^2}{dt^2} V(t) = 8H(\phi) \tag{2}$$

thus by integrating in time

$$V(t) = 4H(\phi)t^2 + V'(0)t + V(0)$$

Assume there exists a t_0, for which $\lim_{t \to t_0} V(t) = 0$. Due to conservation of $|f|^2_{L^2}$, there holds

$$\underbrace{|\Psi|^2_{L^2}}_{const} \leq \frac{2}{d} |\nabla\Psi|_{L^2} \underbrace{|\mathbf{x}\Psi|_{L^2}}_{= V(t) \to 0} \quad \Rightarrow \quad \lim_{t \to t_0} |\nabla\Psi|_{L^2} = \infty$$

Proof 2

$V(0) > 0$, thus the parabola has a root on the positive time axis in any of the three following cases:

- The parabola has a global maximum: $H(\phi) < 0$
- Line with negative inclination: $H(\phi) = 0$ and $V'(0) < 0$
- Angular point on the pos. time axis, with a neg. value

$$V(t) = 4H(t + \frac{V'}{8H})^2 + V - \frac{V'^2}{16H} \;\Rightarrow\; SP = (-\frac{V'}{8H}, V - \frac{V'^2}{16H})$$

$$\Rightarrow \quad V' \leq 0 \quad \text{and} \quad V'^2 \geq 16VH$$

The three cases correspond to the three conditions in the theorem. Remark: For $d\sigma > 2$, there is a $\leq$ in equation (2). Other than that the proof is the same.

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Pseudospectral method to solve the NLS

The NLS is given by

$$i\Psi_t = (\underbrace{-\Delta}_{=:\,L} + \underbrace{|\Psi|^{2\sigma}}_{=:\,NL})\Psi$$

Thus the formal solution to the initial value problem $\Psi(\mathbf{x}, 0) = \phi(\mathbf{x})$ is given by

$$\Psi(\mathbf{x}, t) = e^{-i(L+NL)\delta t} \phi(\mathbf{x}),$$

where $\delta t := t - t_0$. This exponential can be approximated to the second order:

$$e^{-i(L+NL)\delta t} = e^{-iNL\frac{\delta t}{2}} e^{-iL\frac{\delta t}{2}} e^{-iNL\frac{\delta t}{2}} + O(\delta t^2)$$

Skatch of the algorithm

- 1 Solve half the *NL* step in the $\mathbf{x}$ space
- 2 Fouriertransform into k-space
- 3 Solve all of the L-step in the k-space
- 4 Inversfouriertransform back to the $\mathbf{x}$ space
- 5 Solve the other half of *NL* in the $\mathbf{x}$ space

Solution for the nonlinear operator:

$$\Psi(\mathbf{x}, t_0 + \delta t) = \Psi(\mathbf{x}, t_0) e^{-i\,|\Psi(\mathbf{x}, t_0)|^2\,\delta t}$$

Solution for the linear operator:

$$\begin{aligned}
\hat{\Psi}(\mathbf{k}, t_0) &= \mathcal{F}[\Psi(\mathbf{x}, t_0)] \\
\hat{\Psi}(\mathbf{k}, t_0 + \delta t) &= \hat{\Psi}(\mathbf{k}, t_0) e^{-i\,(k_x^2 + k_y^2)\,\delta t} \\
\Psi(\mathbf{x}, t_0 + \delta t) &= \mathcal{F}^{-1}[\Psi(\mathbf{k}, t_0 + \delta t)]
\end{aligned}$$

Contents

Some examples

- Instability of the Towns profile
- Instability of a Gaussian profile
- Nonsymmetric initial conditions

Literature

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