Overview

$$u_t - (\mathcal{L}\, u)_x + p u^{p-1} u_x = 0 \tag{1.1}$$

$$(1.3)\ \begin{cases} c\Phi + \mathcal{L}\,\Phi = \Phi^p \\ \lim_{|x|\to\infty} \Phi(x) = 0 \end{cases} \quad \text{or } (1.5)\ \ [c + v(k)]\,\widehat{\Phi}(k) = \widehat{\Phi^p}(k)$$

Assumption 1.1

$p > 1$, $v(k) \geq 0$, $c > 0$. $\exists$ real analytical solution to in $X = L^2(\mathbb{R}) \cap L^{p+1}(\mathbb{R}) \cap H^{m/2}(\mathbb{R})$

Petviashvili Iteration

$$\widehat{u}_{n+1}(k) = M_n^{\gamma} \frac{\widehat{u_n^p}(k)}{c + v(k)} \tag{1.8}$$

$$M_n = \frac{\int_{-\infty}^{\infty} [c + v(k)][\widehat{u}_n(k)]^2 dk}{\int_{-\infty}^{\infty} \widehat{u}_n(k) \widehat{u_n^p}(k) dk} \tag{1.9}$$

$$\mathcal{H} = c + \mathcal{L} - p\Phi^{p-1}(x) \tag{2.1}$$

Assumption 2.1 on Spectrum of $\mathcal{H}$:

- $\sigma_{L^2}^{\text{discr}}(\mathcal{H})$ for eigenvalues $< c$
- $\sigma_{L^2}^{\text{cont}}(\mathcal{H})$ for eigenvalues $\geq c$
- Nullspace is one-dimensional
- dim. neg. space $n(\mathcal{H}) \geq 1$

Assumption 2.7 :

Either $\Phi^{p-1}(x) \geq 0$ ($\longrightarrow \lambda_{\max}((c + \mathcal{L})^{-1}\,\mathcal{H}) < 1$) or $\lambda_{\max}((c + \mathcal{L})^{-1}\,\mathcal{H}) < 2$

Theorem 2.8

Let $\widehat{\Phi}(k)$ solution to (1.5), assumptions 1.1 and 2.1. Petviashvili Iteration (1.8), (1.9) converges to $\widehat{\Phi}(k)$ in (small) neighbourhood of $\widehat{\Phi}(k)$ if:

1. $1 < \gamma < \frac{p+1}{p-1}$
2. $n(\mathcal{H}) = 1$
3. assumption 2.7 is met.

"If any of the conditions are not met, the Petviashvili iteration diverges from $\widehat{\Phi}(k)$".

Fixed Point Theorem

Let $\mathcal{B}$ a Banach space, $D \subset \mathcal{B}$ open, assume that $A : D \longrightarrow \mathcal{B}$ has fixed point $\bar{f} \in D$, and let A Fréchet diff. in $\bar{f}$ ($A'(\bar{f})$).
$\forall 0 < \varepsilon < 1 - \|A'(\bar{f})\|$ $\quad \exists S(\bar{f}, \delta)$ open such that if $f_0 \in S(\bar{f}, \delta)$:

- The iterates $f_n := A f_{n-1} \in S(\bar{f}, \delta)$
- $\lim f_n = f$
- $\|f_n - \bar{f}\| \leq (\|A'(f)\| + \varepsilon)^n \, \|f_0 - f\|$

Proposition 3.1

$A'(\widehat{\Phi})$ (i.e. Operator (1.8), (1.9) linearized at $\widehat{\Phi}(k)$) has spectral radius smaller than one ($\|A'(\widehat{\Phi})\| < 1$), if

- $1 < \gamma < \frac{p+1}{p-1}$
- $n(\mathcal{H}) = 1$
- assumptions 2.1 and 2.7 are met.

$$X_p := \{U \in L^2 : \langle \Phi^p, U\rangle = 0\}$$

$$q_{n+1}(x) = q_n(x) - (c + \mathcal{L})^{-1}\,\mathcal{H}\,q_n(x) \tag{3.5}$$

Lemma 2.4

$\sigma\left((c+\mathcal{L})^{-1}\mathcal{H}\right)$ in $X_p(\mathbb{R})$ has $n(\mathcal{H}) - 1$ negative eigenvalues.

Lemma 2.5

Positive spectrum of $(c+\mathcal{L})^{-1}\mathcal{H}$ in $X_p(\mathbb{R})$:

1. Infinitely many discrete EV. $0 < \lambda < 1$ (accumulating to 1^-).
2. If $\forall x \in \mathbb{R}$: $\Phi^{p-1}(x) \geq 0$: no EV. > 1.
3. If $\exists x_0 \in \mathbb{R} : \Phi^{p-1}(x_0) < 0$, we also have infinitely many discrete EV. in $1 < \lambda < \lambda_{\max}$ (accumulating to 1^+), and $\lambda_{\max} < 1 + \frac{p}{c} \mid \min_{x\in\mathbb{R}} \Phi^{p-1}(x) \mid < \infty$.

$$\mathcal{H}U = \lambda(c+\mathcal{L})U \tag{2.4}$$

Lemma 2.3

The negative space of $\mathcal{H}$ in $X_p(\mathbb{R})$ has dimension $n(\mathcal{H}) - 1$.

$$\langle \Phi^p, \psi\rangle = 0 \quad \mathcal{H}\psi = \mu\psi - \nu\Phi^p(x) \tag{2.7}$$

$$\psi(x) = \nu\left[\sum_{\mu_k<0} \frac{\langle u_k, \Phi^p\rangle}{\mu - \mu_k} u_k(x) + \sum_{\mu_k>0} \frac{\langle u_k, \Phi^p\rangle}{\mu - \mu_k} u_k(x)\right] \tag{2.8}$$

$$F(\mu) = \frac{1}{\nu}\langle \Phi^p, \psi\rangle = \sum_{\mu_k<0} \frac{|\langle \Phi^p, u_k\rangle|^2}{\mu - \mu_k} + \sum_{\mu_k>0} \frac{|\langle \Phi^p, u_k\rangle|^2}{\mu - \mu_k} \overset{!}{=} 0 \tag{2.9}$$