Additional Calculations

Getting (1.3): $u(x,t) = \phi(x-ct) \xrightarrow{(1.1)} -c\overbrace{\phi'}^{\phi^x} - (\mathcal{L}\phi)_x + p\phi^{p-1}\phi_x = 0$

$$\Updownarrow$$

(1.3): $(c+\mathcal{L})\phi = \phi^p \iff \int dx\left[-c\,\phi_x - (\mathcal{L}\phi)_x + (\phi^p)_x\right] = 0$

Divergence of (1.8), (1.9) without M_n: $\hat{u}_{n+1}(k) = \dfrac{\widehat{u_n^p}(k)}{c+v(k)}$ $\to$ take $u_0 = a\cdot u$, $a>0$

$$\Rightarrow \hat{u}_1 = \frac{a^p\,\widehat{u^p}}{c+v} \overset{(1.3)}{=} a^p\hat{u} \longrightarrow \hat{u}_{n+1} = (a^p)^{n+1}\hat{u} \to \begin{cases} \infty : a>1 \\ \hat{u} : a=1 \ (u_0 = u\,!!) \\ 0 : a<1 \end{cases}$$

Existence of Integrals in (1.8), (1.9): Assume $u_n \in X = L^2 \cap L^{p+1} \cap H^{m/2}$

(1.9):

- $\int_{-\infty}^{\infty} \hat{u}_n\,\widehat{u_n^p}\,dk = \int_{-\infty}^{\infty} u_n u_n^p\,dx = \int_{-\infty}^{\infty} u_n^{p+1}\,dx \overset{(u_n\in L^{p+1})}{<} \infty$
- $\int_{-\infty}^{\infty} c\,\hat{u}_n^2\,dk = c\int_{-\infty}^{\infty} u_n^2\,dx \overset{(u_n\in L^2)}{<} \infty$
- $\int_{-\infty}^{\infty} v(k)\hat{u}_n^2\,dk = \int_{-\infty}^{\infty} (\mathcal{L}u_n)(x)u_n(x)\,dx \overset{\text{P.I.},\ u_n \xrightarrow{|x|\to\infty} 0}{=} \int (\overset{\text{order } \frac{m}{2}}{\mathcal{M}} u_n)(x)(\mathcal{M}u_n)(x)\,dx \overset{u_n\in H^{m/2}}{<} \infty$

Proof of Lemma 1.2: $\gamma \neq 1+2n,\ n\in\mathbb{Z}$: Fix Points of (1.8), (1.9) $\iff$ bound state solution $\hat{\phi}$ of (1.5)

- "$\Leftarrow$": $\hat{u}_n = \hat{\phi} \Rightarrow M_n = \dfrac{\int (c+v)\hat{\phi}\,\hat{\phi}\,dk}{\int \hat{\phi}\,\widehat{\phi^p}\,dk} = \dfrac{\int \widehat{\phi^p}\,\hat{\phi}\,dk}{\int \hat{\phi}\,\widehat{\phi^p}\,dk} = 1$; $\hat{u}_{n+1} = \dfrac{\widehat{\phi^p}}{c+v} \overset{(1.5)}{=} \hat{\phi} = \hat{u}_n$.
- "$\Rightarrow$": Fixpoint u_*: $\int (1.8)\cdot(c+v)\cdot\hat{u}_*\,dk' \Rightarrow \int (c+v)\hat{u}_*^2\,dk' = \int M_*^\gamma\,\widehat{u_*^p}\,\hat{u}_*\,dk' = M_*^\gamma \int \widehat{u_*^p}\,\hat{u}_*\,dk'$

$\hookrightarrow M_* = M_*^\gamma \overset{\gamma\neq 1+2n}{\Rightarrow} M_* = \begin{cases} 0 \xrightarrow{c+v\geq 0} \hat{u}_* = 0 \\ 1 \longrightarrow \hat{u}_* = \hat{\phi} \end{cases}$; $\parallel$ $M_*\int \widehat{u_*^p}\,\hat{u}_*\,dk'$

$\phi' \in$ Nullspace$(\mathcal{H})$: (1.10) $\mathcal{H} = c + \mathcal{L} - p\phi^{p-1}$

$$\int dx\,[\mathcal{H}\phi'] = \int dx\left[(c+\mathcal{L})\phi' - p\phi^{p-1}\phi'\right] = \int dx\left[\{(c+\mathcal{L})\phi\}_x - (\phi^p)_x\right] = (c+\mathcal{L})\phi - \phi^p \overset{(1.3)}{=} 0$$

Continuous spectrum of $\mathcal{H}$: $\mathcal{H} = c + \mathcal{L} - p\phi^{p-1}$: $c > 0$, $v(k) \geq 0$, $p > 1$

2

$\sigma^{cont}(\mathcal{H})$ positive, bounded away from 0: $\exists$ theorem that tells for this case that $\sigma^{cont}(\mathcal{H}) = \sigma^{cont}(c + \mathcal{L})$ $\Rightarrow$ positive, bounded away from 0.

[HP], p. 125, 126: (Fixed Point theorem)

The motivation for the introduction of the Fréchet derivative is that it may be used as the basis of an infinite dimensional calculus. Two typical results are now described. The first, the proof of which must be postponed (see Problem 6.12), is a generalization of the Mean Value Theorem in which the ordinary derivative is replaced by the Fréchet derivative $A'(g)$. Of course here $A'(g)$ is an operator, and the operator norm—that is the norm of $\mathscr{L}(\mathscr{B}, \mathscr{C})$—must be used; similarly the continuity of $A'(\cdot)$ will refer to continuity in the operator norm.

4.4.7 Lemma. *Suppose that $\mathscr{B}$ and $\mathscr{C}$ are Banach spaces. Let D be a convex subset of $\mathscr{B}$ and assume that $A : D \to \mathscr{C}$ is Fréchet differentiable at every point of D. Then*

$$\|Af - Ag\| \leq \|f - g\| \sup_{h \in D} \|A'(h)\|.$$

In other words (cf. (4.3.2)) A satisfies a Lipschitz condition with constant

$$q = \sup_{h \in D} \|A'(h)\|.$$

This bound provides a useful method for estimating the Lipschitz constant in applications of the Contraction Mapping Principle. Evidently the smaller the maximum of $\|A'(h)\|$ on D, the more rapidly will the iterates converge to the fixed point $\bar{f}$. The following says a little more: the asymptotic rate of convergence (that is near $\bar{f}$) is determined by the Fréchet derivative at $\bar{f}$.

4.4.8 Lemma. *Let D be an open subset of the Banach space $\mathscr{B}$, and assume that $A : D \to \mathscr{B}$ has a fixed point $\bar{f}$ in D. Suppose that A is Fréchet differentiable at $\bar{f}$ with $\|A'(\bar{f})\| < 1$. Then given any ε with $0 < \varepsilon < 1 - \|A'(\bar{f})\|$, there is an open ball $S(\bar{f}, \delta)$ such that if $f_0 \in S(\bar{f}, \delta)$, the iterates $f_n = Af_{n-1} (n \geq 1)$ also lie in $S(\bar{f}, \delta)$, $\lim f_n = \bar{f}$, and*

$$\|f_n - \bar{f}\| \leq (\|A'(\bar{f})\| + \varepsilon)^n \|f_0 - \bar{f}\|.$$

Proof. Choose any ε as above. Then from the definition of the Fréchet derivative, there is a $\delta > 0$ such that for any $f \in S(\bar{f}, \delta)$,

$$\|Af - A\bar{f} - A'(\bar{f})(f - \bar{f})\| \leq \varepsilon \|f - \bar{f}\|.$$

Therefore

$$\begin{aligned}
\|Af - \bar{f}\| &= \|Af - A\bar{f}\| \\
&\leq \|Af - A\bar{f} - A'(\bar{f})(f - \bar{f})\| + \|A'(\bar{f})(f - \bar{f})\| \\
&\leq (\|A'(\bar{f})\| + \varepsilon)\|f - \bar{f}\| \\
&\leq \delta.
\end{aligned}$$

That is $Af \in S(\bar{f}, \delta)$. It follows by induction that if $f_0 \in S(\bar{f}, \delta)$ so does f_n for $n \geq 1$. The above inequality with f_n replacing f shows that

$$\|f_{n+1} - \bar{f}\| \leq (\|A'(\bar{f})\| + \varepsilon) \|f_n - \bar{f}\|.$$

Repeated application of this relation gives the final result, from which the convergence follows. □

Th. 2.8:

Continuity of $A'(\hat{u}_n)$ in $S(\hat{\Phi}, \delta_c)$ Proposition 3.4 from [PS]:

Iteration operator (1.8), (1.9), linearized at sequence $\{\hat{\Phi}_n(k)\}_{n=0}^{\infty}$ is continuous in a small open neighbourhood of $\hat{\phi}(k)$.

Proof: See [PS], p. 1118.:

- linearization: similar (a bit more complicated) to calculation for Prop. 3.1
- continuity: Integrals (3.11) exist for $\phi_n \in X \Rightarrow$ " $\sup_{\phi_n} \|A_{lin}(\phi_n)\| < \infty$ "

[HP], p. 116, 117:

4.3.1 Definition. A is said to satisfy a **Lipschitz condition** on D with **Lipschitz constant** q iff there is a $q < \infty$ such that

$$\|Af - Ag\| \leqslant q\|f - g\| \qquad (f, g \in D).$$

In one dimension a function which satisfies a Lipschitz condition is absolutely continuous, and hence differentiable almost everywhere. It is convenient to have the following terminology available when D is unbounded.

4.3.2 Definition. A will be said to satisfy a **local Lipschitz condition** iff for each bounded $S \subset D$, A satisfies a Lipschitz condition on S with Lipschitz constant q_S (which may depend on S).

4.3.3 Definition. A will be called a **contraction** iff it satisfies a Lipschitz condition with Lipschitz constant $q < 1$.

4.3.4 The Contraction Mapping Principle. *Suppose that A maps the closed subset D of the Banach space $\mathscr{B}$ into D and is a contraction. Then A has exactly one fixed point, $\bar{f}$ say, in D. Further, for any initial guess $f_0 \in D$, the successive approximations $f_{n+1} = Af_n$ $(n \geqslant 0)$ converge to $\bar{f}$, and the following estimate for the convergence rate holds:*

$$\|\bar{f} - f_n\| \leqslant q^n(1-q)^{-1}\|Af_0 - f_0\| \tag{4.3.3}$$

Proof. Since A is a contraction,

$$\|f_n - f_{n+1}\| = \|Af_{n-1} - Af_n\| \leqslant q\|f_{n-1} - f_n\|,$$

and it follows from Lemma 1.4.3 that for $n > m$,

$$\|f_m - f_n\| \leqslant q^m(1-q)^{-1}\|Af_0 - f_0\|.$$

This proves that (f_n) is Cauchy. Since D is closed and (f_n) lies in D, (f_n) converges to some $\bar{f}$ say in D. As A is continuous, $A\bar{f} = \lim Af_n = \lim f_{n+1} = \bar{f}$. That is $\bar{f}$ is a fixed point. To prove uniqueness, suppose that $\bar{g}$ is another fixed point. Then

$$\|\bar{f} - \bar{g}\| = \|A\bar{f} - A\bar{g}\| \leqslant q\|\bar{f} - \bar{g}\|,$$

and since $q < 1$, it follows that $\bar{f} = \bar{g}$. □

Calculations for Prop. 3.1:

- (3.3): $w_n(x) = a_n\phi(x) + q_n(x)$ $\quad : q_n \in X_p \Rightarrow \langle w_n - a_n\phi, \phi^p\rangle \overset{!}{=} 0$

 ↳ define a_n : $\langle w_n, \phi^p\rangle = \langle a_n\phi, \phi^p\rangle = a_n\langle\phi, \phi^p\rangle$

$$a_n := \frac{\langle w_n, \phi^p\rangle}{\langle\phi, \phi^p\rangle}$$

- $\underline{m_n = (1-p)a_n}$: (3.2): $\underline{m_n} = (1-p)\dfrac{\int \widehat{\phi^p}\,\hat{w}_n\,dk}{\int\widehat{\phi^p}\,\hat{\phi}\,dk} \overset{(3.3)}{=} (1-p)\dfrac{\int\widehat{\phi^p}(a_n\hat{\phi}+\hat{q}_n)dk}{\int\widehat{\phi^p}\,\hat{\phi}\,dk}$

$$= (1-p)a_n + (1-p)\frac{\overbrace{\int\widehat{\phi^p}\,\hat{q}_n\,dk}^{=\int\phi^p q_n \underset{q_n\in X_p}{=} 0}}{\int\widehat{\phi^p}\,\hat{\phi}\,dk} = \underline{(1-p)a_n}$$

- (3.4): $\underline{m_{n+1} = [p - \gamma(p-1)]m_n}$:

$$\underline{m_{n+1}} \overset{(3.1)}{=} (1-p)\frac{1}{\int\widehat{\phi^p}\hat{\phi}\,dk}\left[\int\widehat{\phi^p}\left(\gamma m_n\hat{\phi} + p\frac{(\widehat{\phi^{p-1}} * \hat{w}_n)(k)}{c+v(k)}\right)dk\right.$$

$$= (1-p)\gamma m_n + p(1-p)\frac{1}{\int\widehat{\phi^p}\,\hat{\phi}\,dk}\underbrace{\int dk\,\frac{\overbrace{\widehat{\phi^p}(k)}^{\hat{\phi}(k)}}{c+v(k)}\int dk'\,\widehat{\phi^{p-1}}(k-k')\hat{w}_n(k')}_{\int dk'\,(\hat{\phi} * \widehat{\phi^{p-1}})(k')\,\hat{w}_n(k')}$$

$$\widehat{\phi\,\phi^{p-1}}(k') = \widehat{\phi^p}(k')$$

$$= (1-p)\gamma m_n + p(1-p)\frac{\overbrace{\int\widehat{\phi^p}(k')\,\hat{w}_n(k')\,dk'}^{m_n}}{\int\widehat{\phi^p}\,\hat{\phi}\,dk} = \underline{(1-p)\gamma m_n + p\,m_n}$$

- (3.5): $\underline{q_{n+1}(x) = q_n(x) - (c+\mathcal{L})^{-1}\mathcal{H}q_n(x)}$: $\Rightarrow \hat{q}_{n+1} = \underbrace{\hat{q}_n - \frac{1}{c+v}[c+v]\hat{q}_n}_{0} + \frac{p}{c+v}\widehat{\phi^{p-1}} * \hat{q}_n$

$$\hat{w}_{n+1} = a_{n+1}\hat{\phi} + \hat{q}_{n+1} \overset{a_n = \frac{m_n}{1-p}}{=} \frac{m_{n+1}}{1-p}\hat{\phi} + \hat{q}_{n+1} \overset{m_{n+1} = [p-\gamma(p-1)]m_n}{=} \underline{\frac{p}{1-p}m_n\hat{\phi}} + \gamma m_n\hat{\phi} + \underbrace{p\frac{\widehat{\phi^{p-1}} * \hat{q}_n}{c+v}}_{(*)}$$

$$(*) = \frac{p}{c+v}\left[\widehat{\phi^{p-1}} * (\hat{w}_n - a_n\hat{\phi})\right] = p\frac{\widehat{\phi^{p-1}} * \hat{w}_n}{c+v} - p\,a_n\hat{\phi} = p\frac{\widehat{\phi^{p-1}} * \hat{w}_n}{c+v} - \underline{p\frac{m_n}{1-p}\hat{\phi}}$$

$$= \gamma m_n\hat{\phi} + p\frac{\widehat{\phi^{p-1}} * \hat{w}_n}{c+v} \quad \blacksquare$$

Iteration Operator: $\hat{u}_{n+1}(k) = M_n^\gamma \dfrac{\widehat{u_n^p}(k)}{c+v(k)}$

$$M_n[\hat{u}_n] = \frac{\int_{-\infty}^{\infty}(c+v(k))[\hat{u}_n(k)]^2 dk}{\int_{-\infty}^{\infty}\hat{u}_n(k)\,\widehat{u_n^p}(k)\,dk}$$

Computation of (3.1), (3.2)

①

5

(a)
- $\frac{1}{f(\cdot)} = \frac{1}{f(0)} + \frac{x}{f(0)^2} \cdot f'(0)$, $\quad \frac{1}{\sum_k a_k \varepsilon^k} = \sum_k b_k \varepsilon^k \to 1 = a_0 \sum_k b_k \varepsilon^k + \varepsilon a_1 \sum_k b_k \varepsilon^k + \dots = a_0 \cdot b_0 + \varepsilon(a_0 b_1 + a_1 b_0) + \mathcal{O}(\varepsilon^2) \Rightarrow \cdot\, b_0 = \frac{1}{a_0}$

  $\cdot\, \frac{a_1}{a_0} = -a_0 b_1 \Rightarrow b_1 = -\frac{a_1}{a_0^2}$

  $= \frac{1}{a_0} - \varepsilon \frac{a_1}{a_0^2} + \mathcal{O}(\varepsilon^2)$

(b)
- $\hat{\Phi}(k)$: $\quad c\,\Phi + \mathcal{L}\Phi = \Phi^p \;\to\; [c+v(k)]\,\hat{\phi}(k) = \widehat{\phi^p}(k)$

$\hat{u}_n(k) = \hat{w}_n(k) + \hat{\phi}(k)$

(c)
- $\hat{w}_0(k) + \hat{\phi}(k) = \hat{u}_0(k) \longrightarrow \hat{w}_0(k) = \hat{u}_0(k) - \hat{\phi}(k) \Rightarrow \hat{w}_n(k) \stackrel{!}{=} \hat{u}_n(k) - \hat{\phi}(k)$

perturbation

$$\hat{w}_{n+1}(k) = \gamma m_n \hat{\phi}(k) + p\,\frac{\widehat{\phi^{p-1}} * \hat{w}_n(k)}{c+v(k)}, \quad m_n = (1-p)\frac{\int_{-\infty}^{\infty}\widehat{\phi^p}(k)\,\hat{w}_n(k)\,dk}{\int_{-\infty}^{\infty}\widehat{\phi^p}(k)\,\hat{\phi}(k)\,dk}$$

What we should get:

$$\boxed{\hat{u}_{n+1}(k) = \hat{w}_{n+1}(k) + \hat{\phi}(k) = \frac{\overbrace{\int_{-\infty}^{\infty}(c+v(s))\left[\hat{w}_n^2(s) + 2\hat{w}_n(s)\hat{\phi}(s) + \hat{\phi}(s)^2\right]ds}^{(1)}}{\underbrace{\int_{-\infty}^{\infty}(\hat{w}_n(s)+\hat{\phi}(s))\,\widehat{(w_n+\phi)^p}(s)\,ds}_{(2)}}\;\underbrace{\left(\frac{\widehat{(w_n+\phi)^p}(k)}{c+v(k)}\right)}_{(0)}}$$

$$\Rightarrow \blacklozenge\,(0)\; \left(\frac{\widehat{(w_n+\phi)^p}(k)}{c+v(k)}\right) = \frac{1}{c+v(k)}\left[\widehat{\left(\phi^p + p\phi^{p-1}w_n + \mathcal{O}(w_n^2)\right)}\right] \underset{\text{FT continuous}}{\leq} \frac{1}{c+v(k)}\left[\widehat{\phi^p} + p\,\widehat{\phi^{p-1}} * \hat{w}_n + \mathcal{O}(w_n^2)\right]$$

$$\blacklozenge\,(1) = \int_{-\infty}^{\infty}(c+v(s))\,\hat{\phi}^2(s) + 2\int_{-\infty}^{\infty}(c+v(s))\,\hat{\phi}(s)\,\hat{w}_n(s)\,ds + \mathcal{O}(w_n^2)$$

$$\blacklozenge\,(2) = \int_{-\infty}^{\infty}\left[\widehat{\phi^p} + p\,\widehat{\phi^{p-1}} * \hat{w}_n + \mathcal{O}(w_n^2)\right](\hat{w}_n(s) + \hat{\phi}(s))\,ds$$

$$= \int_{-\infty}^{\infty}\widehat{\phi^p}(s)\,\hat{\phi}(s)\,ds + \int_{-\infty}^{\infty}\left[\widehat{\phi^p}\,\hat{w}_n(s) + \hat{\phi}\,p\,\widehat{\phi^{p-1}} * \hat{w}_n(s)\right]ds$$

$$\hookrightarrow \frac{1}{(2)} \stackrel{(a)}{=} \frac{1}{\int_{-\infty}^{\infty}\widehat{\phi^p}(s)\,\hat{\phi}(s)\,ds} - \frac{\int_{-\infty}^{\infty}\left[\widehat{\phi^p}\,\hat{w}_n(s) + p\,\hat{\phi}\,\widehat{\phi^{p-1}} * \hat{w}_n(s)\right]ds}{\left(\int_{-\infty}^{\infty}\widehat{\phi^p}(s)\,\hat{\phi}(s)\,ds\right)^2}$$

Compute $(1)\cdot\frac{1}{(2)}\cdot(0)$: (up to $\mathcal{O}(w_n)$)

$$(1)\cdot\frac{1}{(2)}\cdot(0) = \left[\int (c+v(s))\hat\phi^2(s)\,ds + 2\int (c+v(s))\,\hat\phi(s)\,\hat w_n(s)\,ds\right]\cdot\left[\frac{1}{\int \widehat{\phi^p}(s)\,\hat\phi(s)\,ds} + \frac{\int\left[\widehat{\phi^p}\,\hat w_n(s) + p\hat\phi\,\widehat{\phi^{p-1}*\hat w_n}(s)\right]ds}{\left(\int\widehat{\phi^p}(s)\,\hat\phi(s)\,ds\right)^2}\right]\cdot$$

$$\cdot\frac{1}{c+v(k)}\left[\widehat{\phi^p} + p\,\widehat{\phi^{p-1}*\hat w_n}\right]$$

$$= \frac{\widehat{\phi^p}}{c+v(k)}\cdot\int (c+v(s))\,\hat\phi^2(s)\,ds\cdot\frac{1}{\int\widehat{\phi^p}(s)\,\hat\phi(s)\,ds} + \mathcal{O}(w_n)$$

$$(1)\cdot\frac{1}{(2)} = \frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds} + 2\frac{\int (c+v)\,\hat\phi\,\hat w_n\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds} - \frac{\left(\int (c+v)\,\hat\phi^2\,ds\right)\left(\int\widehat{\phi^p}\,\hat w_n + \hat\phi\, p\,\widehat{\phi^{p-1}*\hat w_n}\,ds\right)}{\left(\int\widehat{\phi^p}\,\hat\phi\,ds\right)^2} + \mathcal{O}(w_n^2)$$

need $\left(\frac{(1)}{(2)}\right)^\gamma$ up to $\mathcal{O}(w_n^2)$:

$$\left[(a_0 + a_1\varepsilon)^\gamma = a_0^\gamma + \gamma\, a_0^{\gamma-1}\cdot a_1\varepsilon + \mathcal{O}(\varepsilon^2)\right]$$

$$\left(\frac{(1)}{(2)}\right)^\gamma \overset{\mathcal{O}(w_n^2)}{=} \left[\frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds}\right]^\gamma + \gamma\left[\frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds}\right]^{\gamma-1}\left[2\frac{\int (c+v)\,\hat\phi\,\hat w_n\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds} - \frac{\left(\int (c+v)\,\hat\phi^2\,ds\right)\left(\int\widehat{\phi^p}\,\hat w_n + p\hat\phi\,\widehat{\phi^{p-1}*\hat w_n}\,ds\right)}{\left(\int\widehat{\phi^p}\,\hat\phi\,ds\right)^2}\right]$$

We finally get:

$$\left(\frac{(1)}{(2)}\right)^\gamma\cdot(0) = \frac{\widehat{\phi^p}}{c+v(k)}\cdot\left[\frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds}\right]^\gamma + \frac{p\,\widehat{\phi^{p-1}*\hat w_n}}{c+v(k)}\cdot\left[\frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds}\right]^\gamma$$

$$+ \frac{\widehat{\phi^p}}{c+v(k)}\cdot\gamma\left[\frac{\int (c+v)\,\hat\phi^2\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds}\right]^{\gamma-1}\cdot\left[2\frac{\int (c+v)\,\hat\phi\,\hat w_n\,ds}{\int\widehat{\phi^p}\,\hat\phi\,ds} - \frac{\left(\int (c+v)\,\hat\phi^2\,ds\right)\left(\int\widehat{\phi^p}\,\hat w_n + p\,\widehat{\phi^{p-1}*\hat w_n}\cdot\hat\phi\,ds\right)}{\left(\int\widehat{\phi^p}\,\hat\phi\,ds\right)^2}\right]$$

$$+ \mathcal{O}(w_n^2)$$

Simplify: Use $(c+v)\hat{\phi} = \widehat{\phi^p}$:

$$\frac{\int (c+v)\hat{\phi}^2 \, ds}{\int \widehat{\phi^p}\,\hat{\phi}\, ds} = 1 \quad !$$

hence

$$\left(\frac{(1)}{(2)}\right)^{\gamma} \cdot (0) \overset{O(w_n^2)}{=} \frac{\widehat{\phi^p}}{c + v(k)} + p\,\frac{\widehat{\phi^{p-1}} * \widehat{w_n}}{c + v(k)} + \gamma\,\frac{\widehat{\phi^p}}{c+v(k)} \cdot \frac{2\int (c+v)\hat{\phi}\,\hat{w}_n\, ds - \int \widehat{\phi^p}\,\hat{w}_n\, ds - p\int \widehat{\phi^{p-1}} * \hat{w}_n \cdot \hat{\phi}\, ds}{\int \widehat{\phi^p}\,\hat{\phi}\, ds}$$

$$= \hat{\phi} + p\,\frac{\widehat{\phi^{p-1}} * \widehat{w_n}}{c + v(k)} + \gamma\,\hat{\phi} \cdot \frac{\int \widehat{\phi^p}\,\hat{w}_n\, ds - p\int \widehat{\phi^{p-1}} * \hat{w}_n \cdot \hat{\phi}\, ds}{\int \widehat{\phi^p}\,\hat{\phi}\, ds}$$

We rewrite

$$\int (\widehat{\phi^{p-1}} * \hat{w}_n)(s) \cdot \hat{\phi}(s)\, ds = \iint ds\, dl \;\; \widehat{\phi^{p-1}}(s-l) \cdot \hat{w}_n(l)\, \hat{\phi}(s)$$

$$= \int dl\, (\widehat{\phi^{p-1}} * \hat{\phi})(l) \cdot \hat{w}_n(l) = \int dl \;\; \widehat{\phi^p} \cdot \widehat{w_n}$$

and get for $\hat{w}_{n+1}(k) = \hat{u}_{n+1}(k) - \hat{\phi}(k)$:

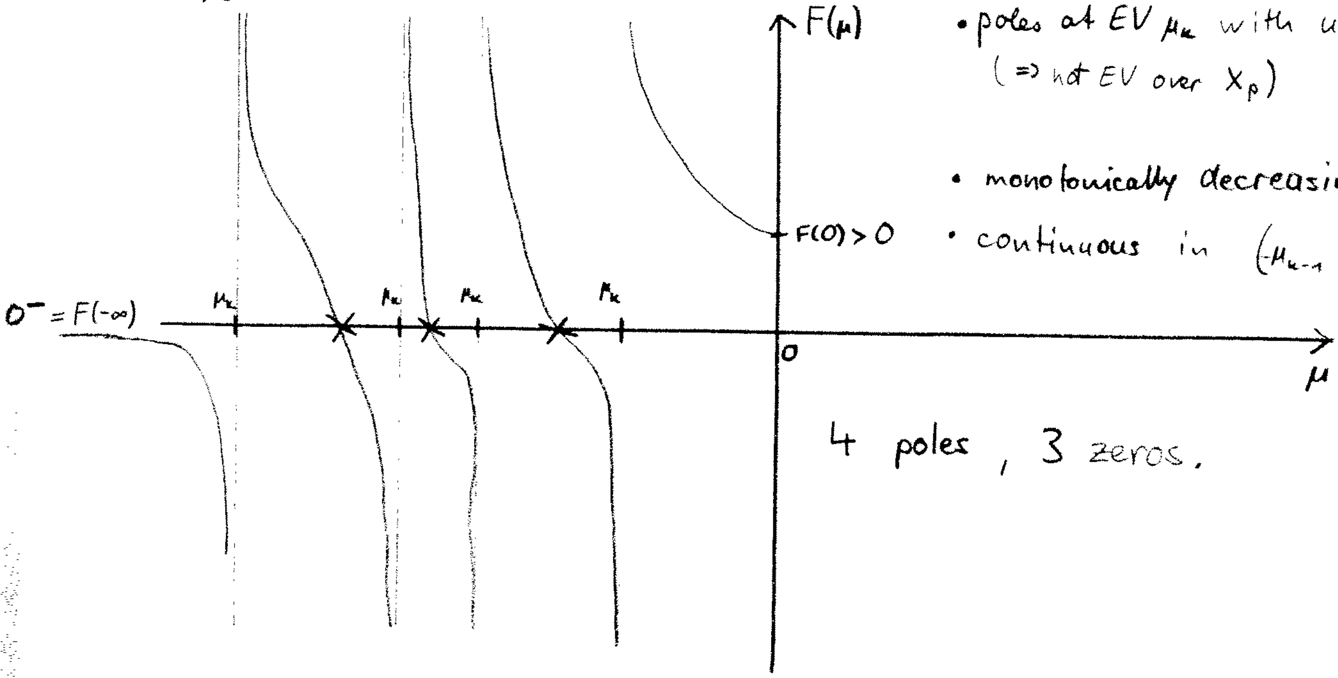
$$\hat{w}_{n+1}(k) = p\,\frac{\widehat{\phi^{p-1}} * \widehat{w_n}}{c + v(k)} + \gamma(1-p)\,\hat{\phi}(k)\,\frac{\int \widehat{\phi^p}\;\widehat{w_n}\, ds}{\int \widehat{\phi^p}\;\hat{\phi}\, ds}$$

Lemma 2.3: Zeros of $F(\mu)$

$$F(\mu) = \sum_{\mu_k<0} \frac{|\langle\phi^p, u_k\rangle|^2}{\mu - \mu_k} + \oint_{\mu_k>0} \frac{|\langle\phi^p, u_k\rangle|^2}{\mu - \mu_k}$$

- continuous at EV μ_k with $u_k \in X_p$ ($\Rightarrow \langle\phi^p, u_k\rangle = 0$)
- poles at EV μ_k with $u_k \notin X_p$ ($\Rightarrow$ not EV over X_p)
- monotonically decreasing!
- continuous in (μ_{k-1}, μ_k)

8



4 poles, 3 zeros.

Lemma 2.5: Form (2.12): $(c+\mathcal{L})U - \frac{p}{1-\lambda}\phi^{p-1}(x)U = 0$

(2.4): $\mathcal{H}U = \lambda(c+\mathcal{L})U \to (c+\mathcal{L})U(1-\lambda) - p\phi^{p-1}U = 0 \Leftrightarrow (c+\mathcal{L})U - \frac{p}{1-\lambda}\phi^{p-1}U = 0$

$$(c+\mathcal{L})U - p\phi^{p-1}U = \lambda(c+\mathcal{L})U$$