Theory and Numerics of Solitary Waves

- introductory lecture -

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Korteweg-de Vries equation $u_t + uu_x + u_{xxx} = 0$

kdv_1soliton.mpg
http://www.ma.hw.ac.uk/solitons

kdv-2soliton.mpg
http://people.uncw.edu/hermanr/research/solitons.htm

But before KdV was J.S. Russell and his “Wave of Translation”
- quotation from John Scott Russell, 1834
http://en.wikipedia.org/wiki/Wave_of_translation.htm

Russell's wave

http://www.ma.hw.ac.uk/solitons/soliton1b.html

Tidal Bore

http://www.severn-bore.co.uk

Tidal Bore and Internal Waves

Strait of Gibraltar - Camarinal Sill
http://earthobservatory.nasa.gov/Newsroom/NewImages/images.php3?img_id=16581
- discussion of internal waves: Ablowitz and Segur, 1981, Sec. 4.1.b

Morning Glory

http://www.dropbears.com/brough/gallery/MG2005/index.htm

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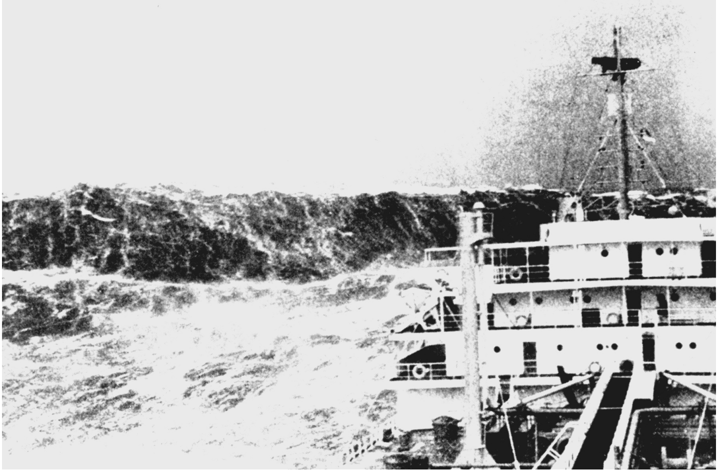
Freak Waves / Rogue Waves / Killer Waves / Extreme Waves



http://www.photolib.noaa.gov/historic/nws/wea00800.htm
Bay of Biscay, France
http://en.wikipedia.org/wiki/Freak_wave

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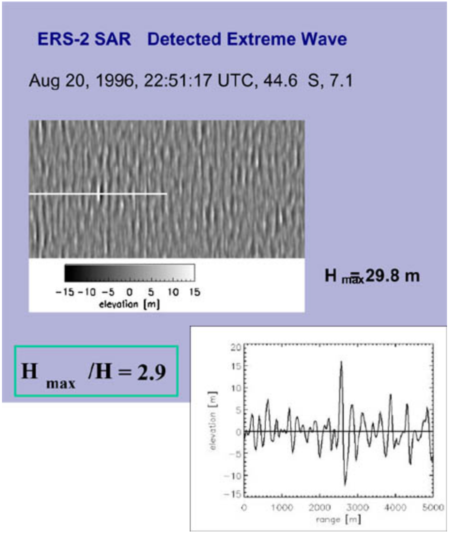

REFERENCE FOR THE FIGURE ??

Brief history of solitons and completely integrable systems

- 1843 - J.S. Russell, Edinburgh-Glasgow canal
  - witnessed a shallow water soliton
  - experimentally deduced $c^2 = g(h+a)$, where h is the water depth and a is the seoliton amplitude
- 1895 Korteweg - de Vries equation (KdV) proposed

$$u_t + uu_x + u_{xxx} = 0$$

- 1955 - Fermi, Pasta and Ulam (LANL)

$$\ddot{x}_n = x_{n+1} - 2x_n + x_{n-1} + \alpha \left[(x_{n+1} - x_n)^p - (x_n - x_{n-1})^p \right], \quad p = 2, 3$$

  - numerical study of weakly nonlinearly (quadratically and cubically) coupled oscillator chains
  - expected equipartition of energy (among all linear modes) caused by the nonlinearity
  - obtained recurrence after some time, e.g. $T \approx 1000 T_1, T_1 = 2\pi/\omega_1, \omega_1 \ldots$ frequency of the fundamental lin. mode

Brief history of solitons and completely integrable systems - cont.

- 1965 - Zabusky and Kruskal
  - numerical study of KdV with periodic BC with $u(x,0) = \cos(\pi x)$
  - obtained steepening and evolution into a pulse train (taller pulses faster)
  - elastic interactions of the pulses, branded the name **soliton**
- 1967 - Gardner, Greene, Kruskal and Miura
  - Inverse Scattering Transform (IST) for KdV
- 1968 - P. Lax
  - generalization of the IST $\rightarrow$ Lax pair
- 1972 - Zakharov and Shabat
  - IST for the Nonlinear Schrödinger equation
- 1973 - Ablowitz, Kaup, Newell and Segur
  - further generalization of the IST
- ...

Solitary waves:

- waves of permanent form
- localised (approach zero or another constant at infinity)

Solitons = solitary waves which

- interact strongly with other solitons and emerge from the collision unchanged apart from a phase and position shift
- no necessary and sufficient condition for soliton existence
- solitary waves arise due to perfect balance between nonlinearity - steepening (water), focusing (light), ... - and dispersion
- often (not always) PDEs with soliton solutions are "completely integrable"
- complete integrability:
  - no necessary and sufficient condition (unlike in Hamiltonian ODEs)
  - necessary cond.: infinite number of conserved quantities
  - usual method: inverse scattering transformation (IST)
  - solution of a system integrable via IST is a soliton $\Leftrightarrow$ its IC generates only point spectrum in the direct scattering problem

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Soliton equation examples:

1. Korteweg de Vries (KdV) equation (shallow water waves, internal waves, ...)

$$u_t + uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, t \geq 0$$

2. Sine Gordon equation (Josephon junctions, self-induced transparency, ...)

$$u_{tt} - u_{xx} + \sin(u) = 0$$

3. Cubic Nonlinear Schrödinger (NLS) equation (pulses in optical fiber, deep water waves, ...)

$$iu_t + u_{xx} + |u|^2 u = 0$$

4. Massive Thirring model

$$i(u_t + u_x) + v + |v|^2 u = 0$$
$$i(v_t - v_x) + u + |u|^2 v = 0$$

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Solitary wave (but not soliton) equation examples:

1. Cubic-quintic NLS (plasma waves, fiber lasers)

$$iu_t + u_{xx} + |u|^2 u - \gamma |u|^4 u = 0$$

2. NLS with saturable nonlinearity (pulses in optical fibers with saturable nonlinearity)

$$iu_t + u_{xx} + \frac{|u|^2}{1+\gamma|u|^2} = 0$$

3. generalized KdV (has solitary waves of compact support = “compactons”)

$$u_t + (u^n)_x + (u^n)_{xxx} = 0$$

4. Coupled Mode Equations (light pulses in fiber Bragg gratings)

$$i(u_t + u_z) + \kappa v + (|u|^2 + 2|v|^2)u = 0$$
$$i(v_t - v_z) + \kappa u + (|v|^2 + 2|u|^2)v = 0$$

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$$\begin{aligned} i(u_t + u_z) + \kappa v + (|u|^2 + 2|v|^2)u &= 0 \\ i(v_t - v_z) + \kappa u + (|v|^2 + 2|u|^2)v &= 0 \end{aligned}$$