Oblique dark solitons and dispersive shock waves in supersonic flow of a Bose-Einstein condensate past an obstacle

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Motivation

- **Recent experiments on generation and propagation of dispersive shock waves in Bose-Einstein condensates (BECs):**

  E. A. Cornell (2004) http://jilawww.colorado.edu/bec/papers.html

  M.A. Hoefer, M.J. Ablowitz, I. Coddington, E.A. Cornell, P. Engels, & V. Schweikhard, Phys. Rev. A (2006).

  I. Carusotto, S.X. Hu, L.A. Collins, and A. Smerzi, Phys. Rev. Lett. (2006).

  and in photorefractive crystals:

  W. Wan, S. Jia & J.W. Fleischer, Nature Physics (2007).

- **Earlier theoretical works on supersonic flow past slender bodies in dissipationless weakly dispersive media**

  A.V. Gurevich, A.L. Krylov, V.V. Khodorovskii and GE, Journ Exp Theor Phys (1995, 1996)

Overview

- **Introduction: Gross-Pitaevskii equation and quantum hydrodynamics**
- **Supersonic flow of a BEC past an obstacle:**
  - **experimentally observed patterns**
  - **numerical simulation**
  - **analytic solutions : oblique dark solitons and “ship waves”**
- **Reconciliation: semi-classical description of oblique dispersive shock waves in highly supersonic flow of a BEC past slender body**
- **Conclusions**

Gross-Pitaevskii equation

Condensate of Bose-atoms in the mean-field approximation is described by the complex "order parameter" Ψ which satisfies the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + g|\Psi|^2\Psi + V(\mathbf{r})\Psi,$$

where $V(\mathbf{r})$ is the potential of external forces (magnetic trap or moving impurity); g is an effective coupling constant ($g > 0$: repulsive interactions, defocussing);

Normalisation:

$$\int |\Psi|^2 d\mathbf{r} = N,$$

where N is the number of atoms in the condensate (typically $\sim 10^6 - 10^7$).

The multi-dimenional NLS equation in an external potential: **nonlinear matter waves**

The NLS equation with defocussing: some highlights

- Integrable in 1D, not integrable in multidimensions.
- Has dark soliton solutions stable in 1D, unstable in 2D and 3D.
- Is a model equation for the description of the envelope of electric field in nonlinear optics: an important parallel with BEC physics, which is widely known but only very recently utilised in experiments on **all-optical modelling of the shock wave propagation through a BEC** :

  Wan, Jia & Fleischer, Nature Physics (2007) – optical shocks in photorefractive crystals – NLS with saturable nonlinearity.

Irrotational quantum hydrodynamics

We introduce the (Madelung) substitution

$$\Psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\exp\left(\frac{i\Theta(\mathbf{r},t)}{\hbar}\right), \qquad \mathbf{u} = \nabla\Theta\,,$$

where $n(\mathbf{r},t)$ is the density of atoms in a BEC and $\mathbf{u}(\mathbf{r},t)$ denotes its velocity field

After the passage to dimensionless variables we arrive at the reduction of the GP equation in the form of irrotational **"quantum hydrodynamics"**

$$\frac{1}{2}n_t + \nabla\cdot(n\mathbf{u}) = 0,$$

$$\frac{1}{2}\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla n + \nabla V(\mathbf{r}) + \nabla\left[\frac{(\nabla n)^2}{8n^2} - \frac{\Delta n}{4n}\right] = 0\,,$$

$$\nabla\times\mathbf{u} = 0\,.$$

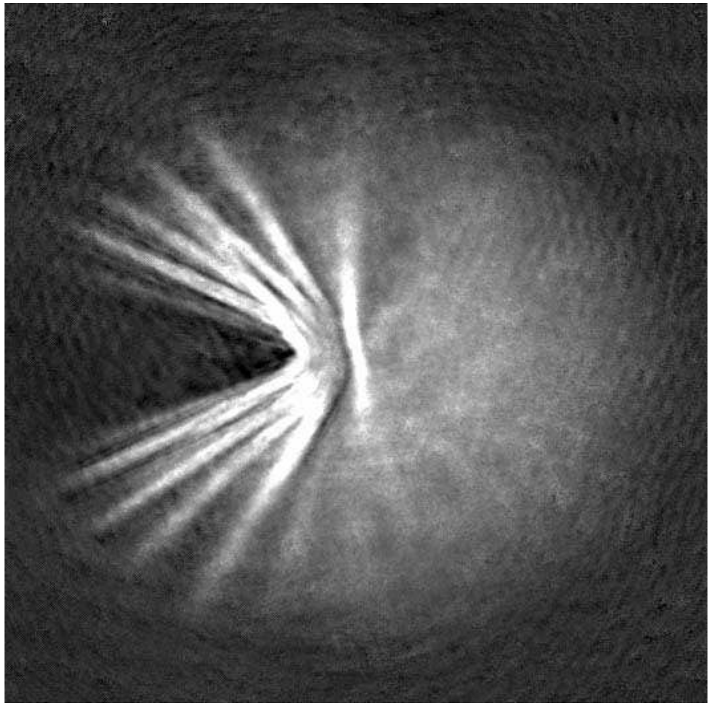
The structure:

irrotational shallow water + external potential + quantum pressure (dispersion)

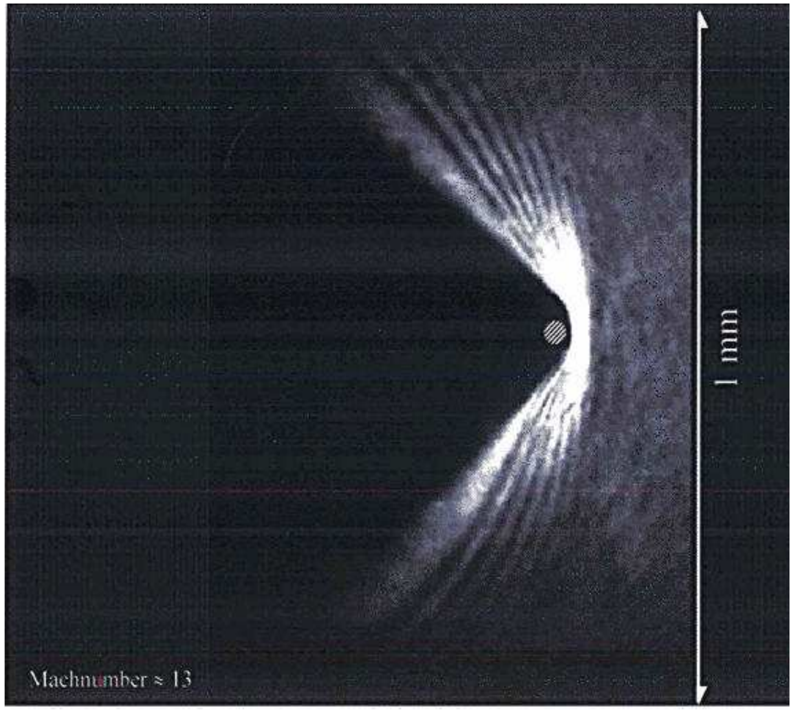
Supersonic flow past an obstacle: experiment

E. Cornell (JILA, Boulder, Colorado, 2004) http://jilawww.colorado.edu/bec/papers.html

Radial expansion of a BEC after switching off the transverse trap potential. The local Mach number $M = u/c_s \sim r/r_0$ — asymptotically does not depend on t for $t \gg 1$.



Solitons?

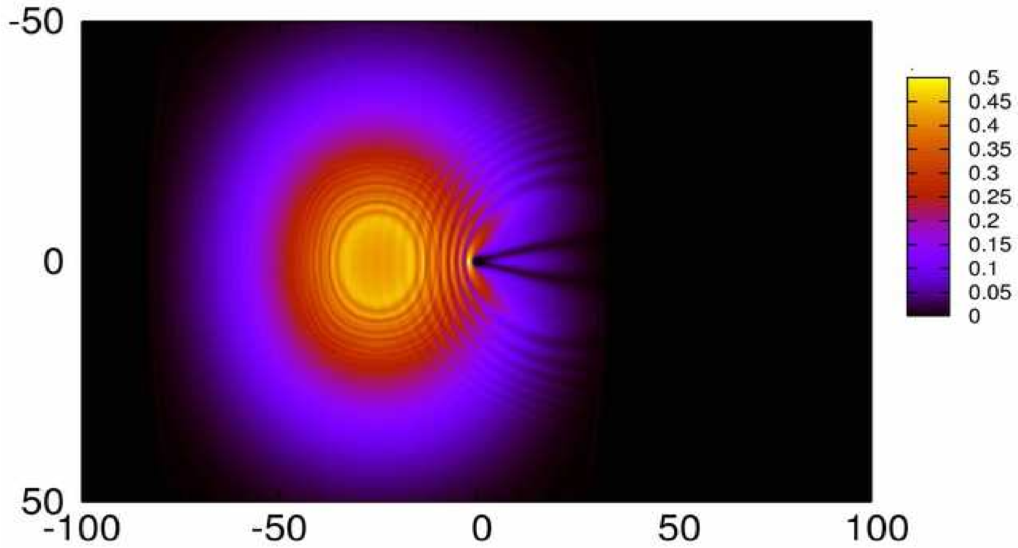


Linear wave radiation?

Supersonic flow past an obstacle: numerical simulation

GE, A. Gammal & A.M. Kamchatnov, Nucl Phys A (2007)

Density plot. Obstacle: impenetrable disc of radius $r = 1$ placed at $(0, 0)$ in a BEC radially expanding from the centre at $(-25, 0)$. Initial radius $R = 25$; $t = 8$.



- Bow waves (small-amplitude ripples)
- V-shaped dark trace behind the obstacle: what are they?

Identification of the wave patterns generated in supersonic flow of a BEC past an obstacle

For $M > 1$ there are two dominant mechanisms of energy transfer between a moving object and a BEC: **vortex formation** and **sound radiation**. So the bow wave pattern can be naturally identified with sound waves and the V-shaped trace with "vortex streets" (aligned vortex-antivortex pairs).

T. Winiecki, B. Jackson, J.F. McCann, and C.S. Adams, (1999, 2000)

Are there dark solitons at all?

Reasoning: classical gas-dynamics problem of supersonic steady 2D flow past body – formation of shock waves. Dispersion instead of dissipation – formation of solitons:

Supersonic flow past slender body in the KdV approximation:

- *Karpman, JETP Lett (1967)*
- *Mei, J Fluid Mech(1968)*
- *Gurevich, Krylov, Khodorovskii & El, Journ Exp Theor Phys (1995, 1996)*

Dark solitons in a BEC

General paradigm: 1D and 2D dark solitons in a BEC are **inherently unstable** with respect to small 2D and 3D perturbations respectively:

“Snake” instability, decay into vortex-antivortex pairs (apparently consistent with the identification of the V-shaped traces behind the obstacle with vortex streets by Winiecki et al. 1999)

- E.A. Kuznetsov and S.K. Turitsyn (1982) - general theory
- A.V. Mamaev, M. Saffman and A.A. Zozulya, *Phys. Rev. Lett.* (1996) - nonlinear optics
- B.P. Anderson et. al., *Phys. Rev. Lett.* (2001). - BEC
- and many more...

This instability statement, however, applies to certain configurations. What about supersonic flow of a BEC past an obstacle?

1. Oblique dark solitons in supersonic flow of a BEC

GE, A. Gammal & A.M. Kamchatnov, Phys. Rev. Lett. (2006).

2D stationary irrotational quantum hydrodynamics equations for the density $n(x, y)$ and two components of the velocity field $\mathbf{u} = (u(x, y), v(x, y))$:

$$\begin{aligned} (nu)_x + (nv)_y &= 0, \\ uu_x + vu_y + n_x + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_x &= 0, \\ uv_x + vv_y + n_y + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_y &= 0\,, \\ u_y - v_x &= 0\,. \end{aligned} \tag{1}$$

Boundary conditions at infinity

$$n \to 1, \quad u \to M > 0, \quad v \to 0 \quad \text{as} \quad |x| + |y| \to \infty.$$

Bernoulli integral:

$$\frac{1}{2}(u^2+v^2)+n+\frac{1}{8n^2}(n_x^2+n_y^2)-\frac{1}{4n}(n_{xx}+n_{yy})=\frac{M^2}{2}+1.$$

Looking for the “travelling wave” solution

$$n=n(\theta),\quad u=u(\theta),\quad v=v(\theta),$$

where $\theta = x - ay$, and a denotes a slope of the wave crests with respect to y-axis.

Then, after a chain of elementary integrations, one obtains

$$u=\frac{M(1+a^2n)}{(1+a^2)n},\quad v=-\frac{aM(1-n)}{(1+a^2)n},$$

$$n(\theta)=1-\frac{1-p}{\cosh^2[\sqrt{1-p}\,\theta/\sqrt{1+a^2}]},$$

where

$$p=\frac{M^2}{1+a^2},$$

– **a stationary oblique dark soliton** with the slope a with respect to y-axis.

Is this solution physical? What about stability?

2D supersonic uniform flow past an obstacle: numerical simulation

Let the order parameter ψ in the Gross-Pitaevskii equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(\mathbf{r})\psi + |\psi|^2\psi$$

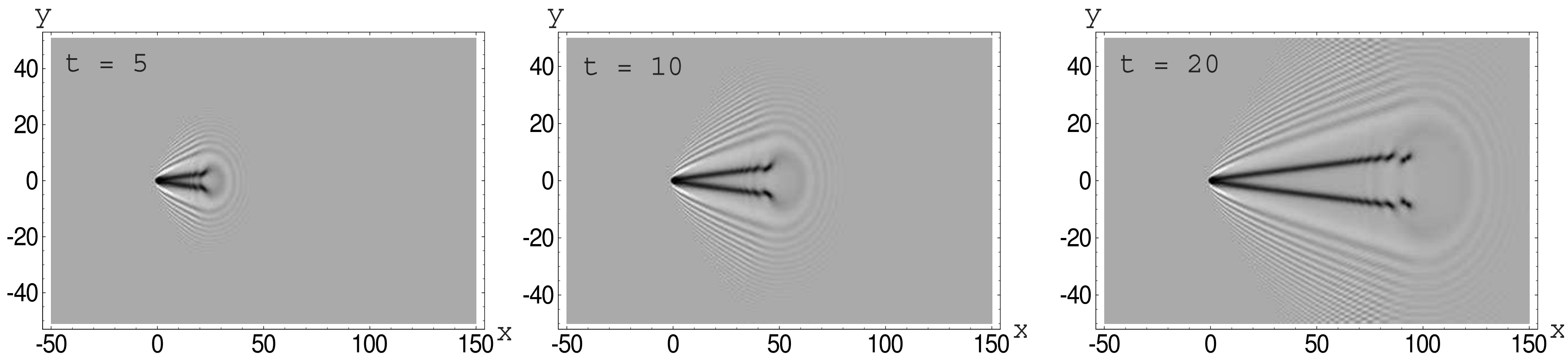
satisfy the initial condition

$$\psi(x,y)|_{t=0} = \exp(iMx),$$

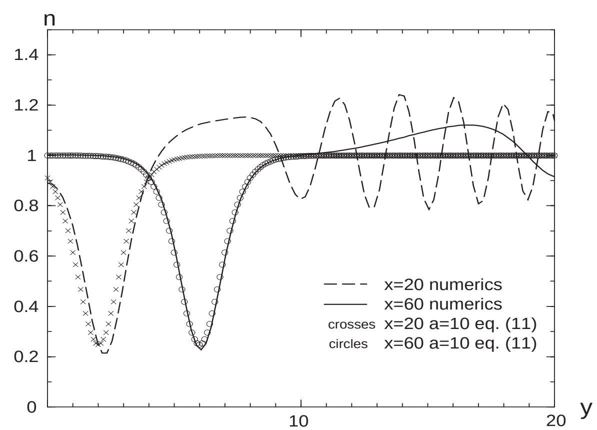
that is the BEC flow with density $n = 1$ and velocity $u = M$ is “switched on” at $t = 0$.

The potential $V(x, y)$ corresponds to the interaction of the condensate with the obstacle which is modelled by an impenetrable disk with radius $r = 1$.

Numerics: $r = 1,\ M = 5$



Comparison of the numerical profiles of density with the formula for the oblique dark soliton.

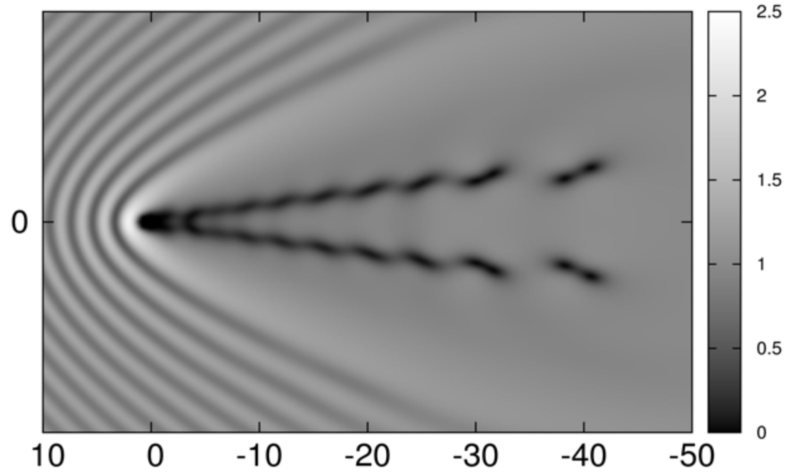


The oblique dark solitons indeed persist in the supersonic BEC flow past an obstacle.

Vortex streets at free ends of solitons!

Closer look: vortex street – oblique dark soliton dynamics

Kamchatnov, Gammal and GE, in progress

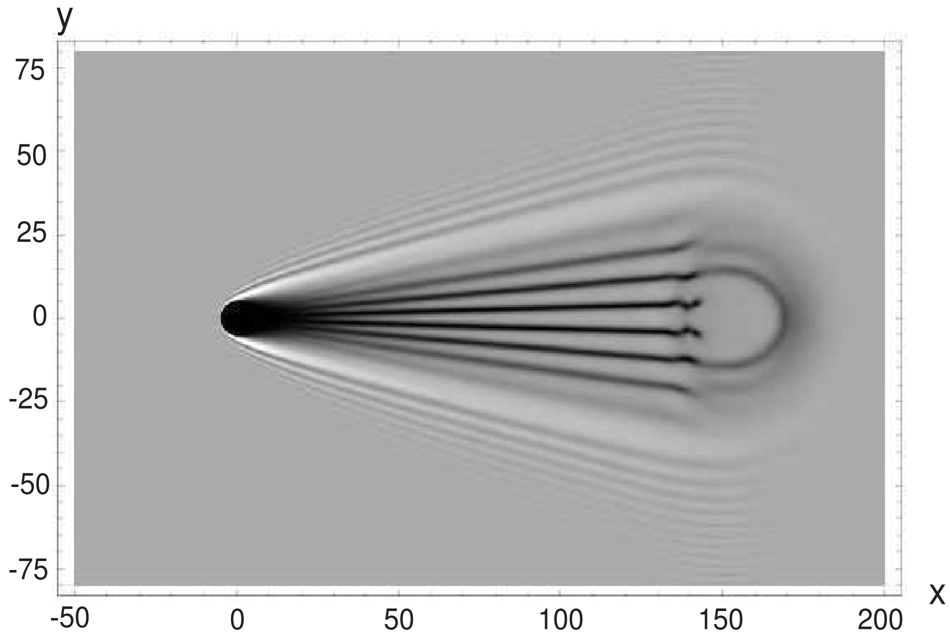


$r = 1,\ M = 2,\ t = 25$: **emission of vortex – antivortex pairs** : **vortex street**

Observation: for sufficiently large M the rate of emission of vortex - antivortex pairs is greater than their separation rate so they gradually get "trapped" in the oblique dark soliton profile with certain slope a: the process opposite to decay of a dark soliton into vortex-antivortex pairs!

Increasing the size of the obstacle

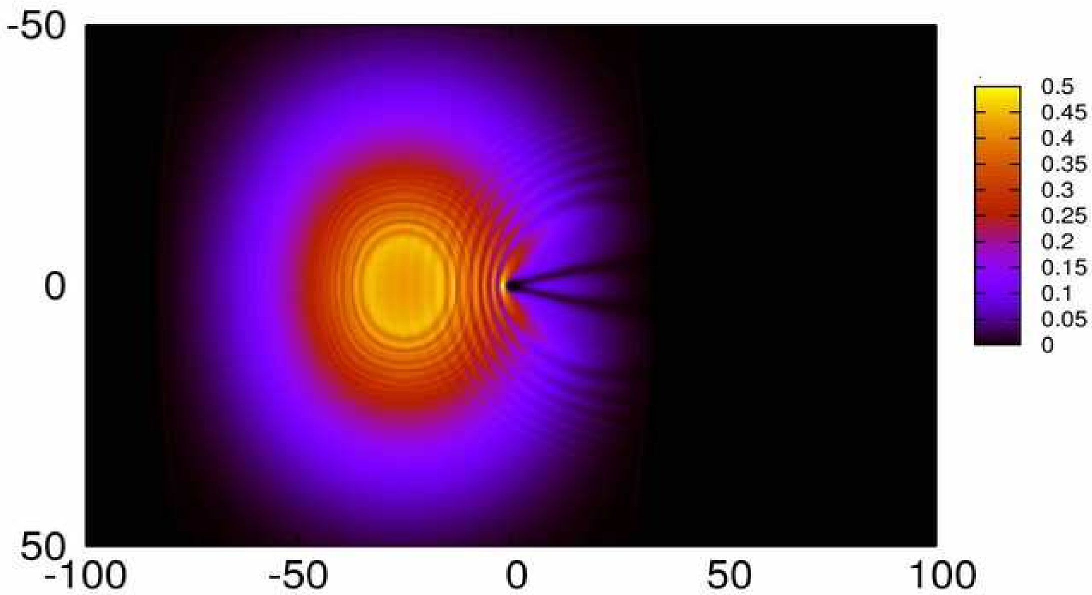
$r = 5$, $M = 5$, $t = 30$: “fan” of oblique dark solitons



Oblique dark solitons serve as “building blocks” in wave patterns occurring in supersonic BEC flow past obstacles.

Question: Amplitudes, slopes and number of oblique solitons in terms of M and r?

2. Bow ripples



Radiation of "ship waves"

Yu.G. Gladush, GE, A. Gammal, and A.M. Kamchatnov, Phys Rev A (2007)

Kelvin's theory applied to the GP equation.

Linear dispersion relation for 2D upwind stationary waves in a BEC

$$n - 1 \sim u - M \sim v \sim \exp(i(k_x x + k_y y)), \quad k_x < 0:$$

$$G(k_x, k_y) \equiv M k_x + k\sqrt{1 + \frac{k^2}{4}} = 0.$$

The wave phase

$$\theta = \int_0^{\mathbf{r}} \mathbf{k} \cdot d\mathbf{r}\,,$$

where

$$\frac{\partial k_x}{\partial y} - \frac{\partial k_y}{\partial x} = 0, \qquad (2)$$

Centred solution of (2)

$$\frac{y}{x} = -\frac{\partial G / \partial k_y}{\partial G / \partial k_x}.$$

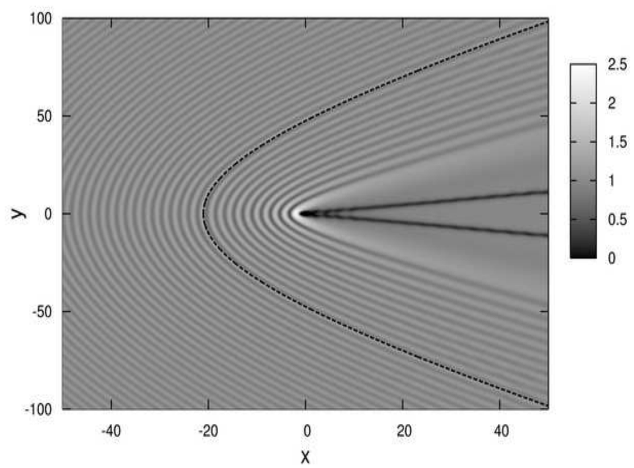
Lines of constant phase θ are specified parametrically in Cartesian coordinates

$$x = \frac{4\theta}{k^3} \cos\eta(1 - M^2 \cos 2\eta), \quad y = \frac{4\theta}{k^3} \sin\eta(2M^2 \cos^2\eta - 1). \tag{3}$$

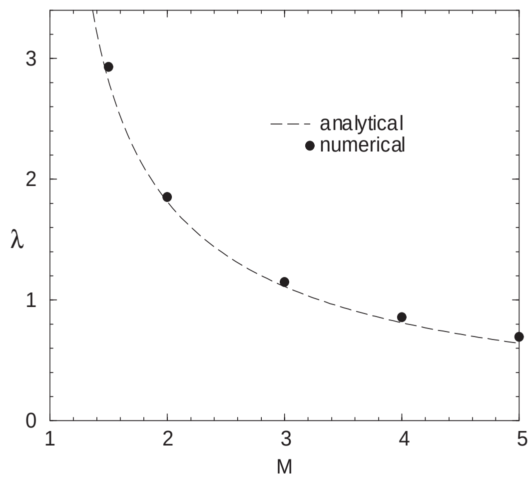
$$-\sqrt{M^2 - 1} \le \frac{y}{x} \le \sqrt{M^2 - 1}.$$

Numerical simulation $M = 2$, $r = 1$.

Dashed line : analytical solution (3)



Comparison for the wavelength λ at $y = 0$

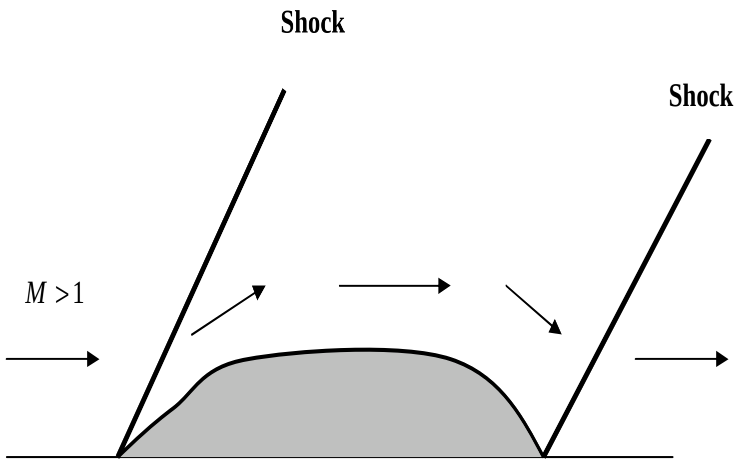


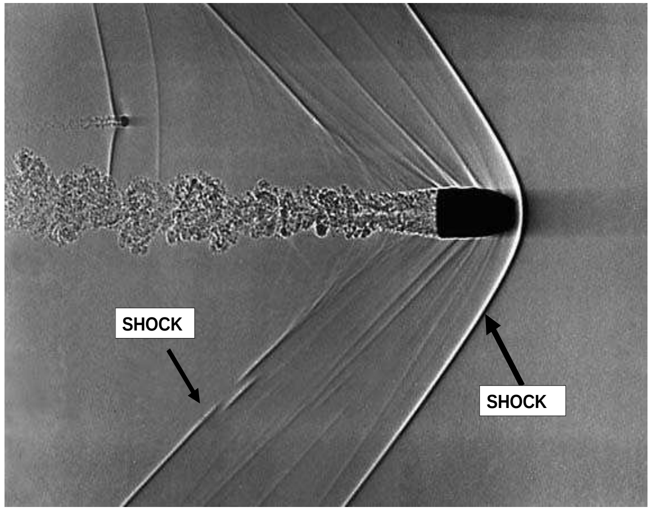
- "Ship waves" are generated <u>outside the Mach cone</u>
- Linear theory works well for reasonably large values of M.

Reconciliation of different wave patterns in the framework of hypersonic flow past slender body problem

GE and A.M. Kamchatnov, Phys Lett A (2006)

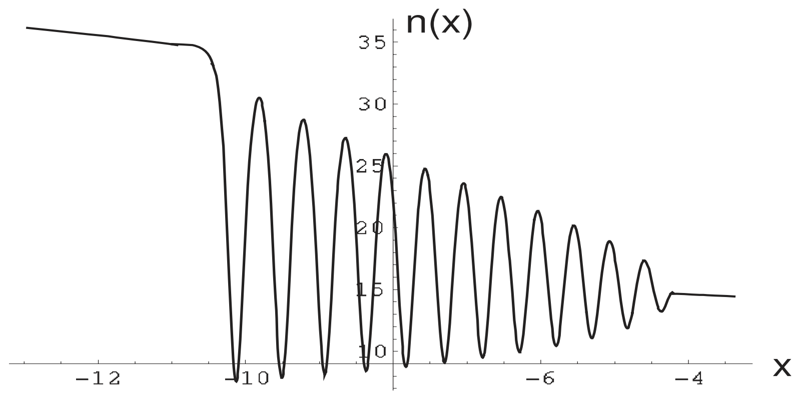
- **Classical dissipative gas dynamics: formation of two shocks (oblique jumps of compression).**





- **Dispersion instead of dissipation: resolution of shocks into expanding nonlinear wave structures – dispersive shock waves**

Oscillatory profile of the dispersive shock wave



Modulated periodic solution of the GP equation. The oscillatory zone expands with y.

Asymptotic description: Gurevich-Pitaevskii (1974) approach in the framework of the Whitham modulation theory.

- **General theory** of dispersive shock waves in 1+1 NLS equation : A. Gurevich & A. Krylov (1987), GE & A. Krylov (1995), G. Biondini & Yu. Kodama (2006)
- **Blast waves in BECs**: A. Kamchatnov et al (2004), M. Hoefer et al (2006)
- Dispersive shock waves in **photorefractive crystals**: 1+1 NLS equation with **saturable nonlinearity**: GE, A. Gammal, E. Khamis, R. Kraenkel & A. Kamchatnov (2007)

2D steady irrotational NLS flow past a large obstacle

2D steady flow $\partial/\partial t \equiv 0, \;\; n = n(x,y), \;\; \mathbf{u} = (u(x,y), v(x,y))$

Obstacle profile: $y = F(x) > 0, \;\; F(0) = 0, \;\; F(L) = 0, \;\; |F'(x)| < \infty, \;\; L \gg 1.$

Governing equations :

$$(nu)_x + (nv)_y = 0, \qquad u_y - v_x = 0\,,$$

$$uu_x + vu_y + n_x + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_x = 0,$$

$$uv_x + vv_y + n_y + \left(\frac{n_x^2 + n_y^2}{8n^2} - \frac{n_{xx} + n_{yy}}{4n} \right)_y = 0.$$

Boundary conditions:

Impenetrability at body surface : $\quad v = uF'(x) \;\;$ on $\;\; y = F(x).$

Uniform flow at infinity : $\quad n \to 1, \;\; u \to M, \;\; v \to 0 \;\;$ as $\;\; |x| + |y| \to \infty,$

Hypersonic reduction, $M \gg 1$ (paraxial approximation)

Asymptotic expansions

$$u = M + u_1 + O(1/M), \qquad T = x/2M, \quad Y = y, \quad M \gg 1\,.$$

Then to leading order the 2+0 GP equation reduces to

$$\begin{aligned} \tfrac{1}{2}n_T + (nv)_Y &= 0, \\ \tfrac{1}{2}v_T + vv_Y + n_Y + \left(\frac{n_Y^2}{8n^2} - \frac{n_{YY}}{4n}\right)_Y &= 0. \end{aligned} \tag{4}$$

System (4) represents the hydrodynamic form of 1+1 **defocusing NLS equation**

$$i\phi_T + \phi_{YY} - 2|\phi|^2\phi = 0$$

for the complex field variable $\phi(Y,T) = \sqrt{n}\exp(i\int^Y v(Y',T)dY')$

Transformation of boundary conditions

Under the hypersonic transformation

$$u = M + u_1 + O(1/M), \qquad T = x/2M, \quad Y = y, \quad M \gg 1 \ :$$

- **Impenetrability** condition at the body surface $y = F(x)$ asymptotically as $M \to \infty$ transforms into the kinematic condition at a **"piston"**, where the motion of the piston is specified by the body surface function

$$v = \frac{1}{2}\frac{df}{dT} \quad \text{at} \quad Y = f(T)\,, \quad \text{where} \ \ f(T) = F(2MT)$$

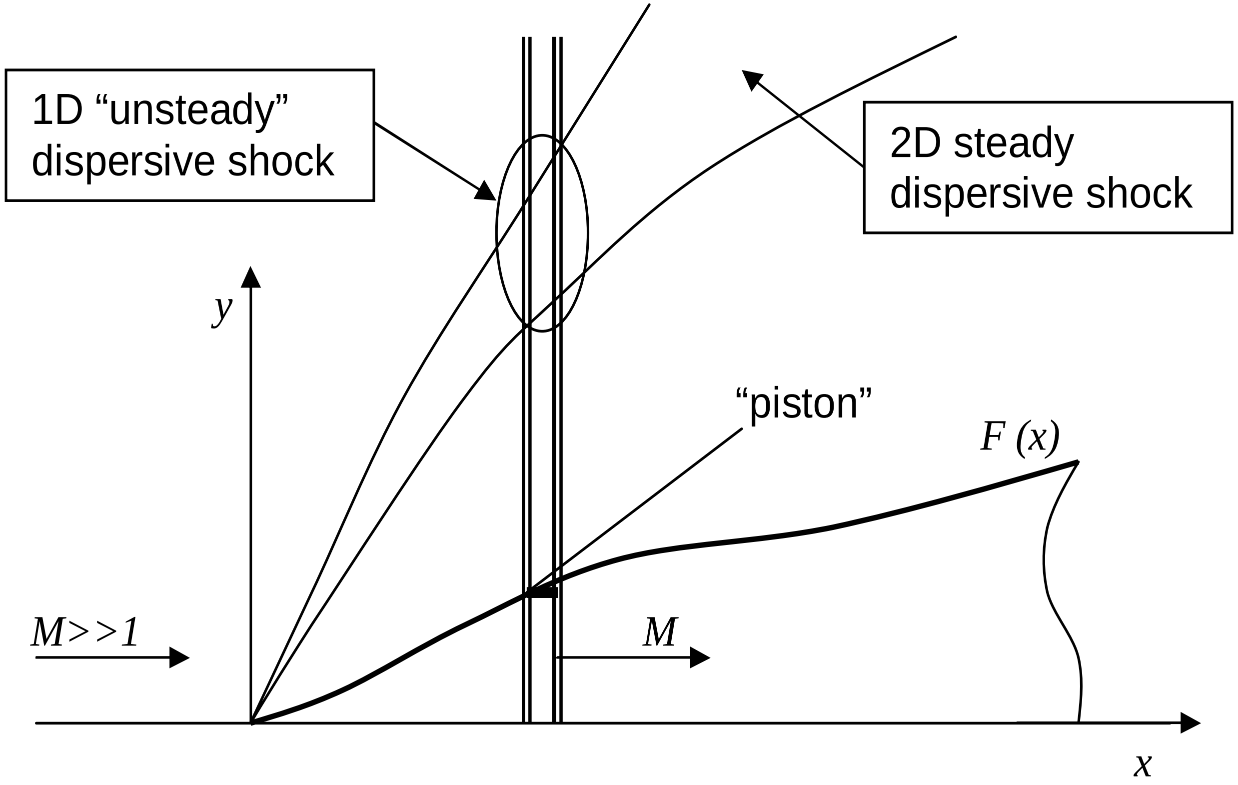
provided $\alpha M = \mathcal{O}(1)$ where $\alpha = \max|F'(x)| \ll 1$ – slender body

- Condition of the **uniform flow at infinity** becomes:

$$n \to 1\,, \quad v \to 0\,, \quad \text{as} \quad Y \to \infty\,.$$

Generation of a dispersive shock in 1+1 dispersive hydrodynamics due to piston motion.

Piston analogy for the hypersonic flow past body in dispersive media



The piston problem for the NLS equation:

$$i\phi_T + \phi_{YY} - 2|\phi|^2\phi = 0\,, \qquad \phi = \sqrt{n}\,\exp(i\int^Y v(Y',T)dY')$$

$$v(f(T),T) = f'(T)/2\,, \qquad n(Y \to \infty) \to 1\,, \quad v(Y \to \infty) \to 0\,.$$

An equivalent initial-value problem

$$n(Y,0) = ?\,, \qquad v(Y,0) = ?$$

The piston problem for the NLS equation: semi-classical reformulation

Let $T_0 = L/(2M) \gg 1$ be a large parameter (long body) so that $f(T) \to f(T/T_0)$.

Introduce

$$\epsilon = 1/T_0 \ll 1\,, \quad Y' = \epsilon Y, \quad T' = \epsilon T$$

and consider the same piston problem. Then, on omitting primes we have

$$i\epsilon\phi_T + \epsilon^2\phi_{YY} - 2|\phi|^2\phi = 0, \quad \epsilon \ll 1$$

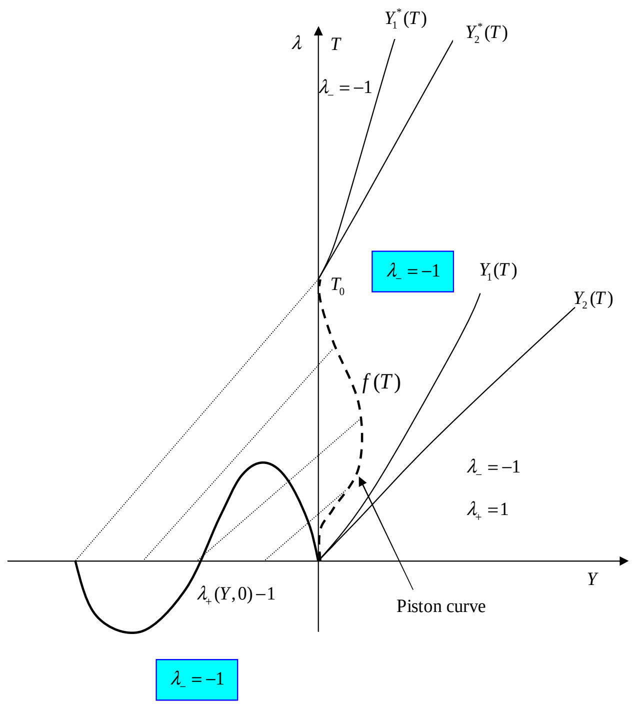
$$v(f(T),T) = f'(T)/2\,, \qquad n(Y \to \infty) \to 1\,, \quad v(Y \to \infty) \to 0\,.$$

In the semi-classical limit, as $\epsilon \to 0$, the solution, outside the dispersive shock wave regions, is described by the (1+1) shallow-water equations (the dispersionless limit of the NLS equation)

$$\frac{\partial \lambda_\pm}{\partial T} + V_\pm(\lambda_+, \lambda_-)\frac{\partial \lambda_\pm}{\partial Y} = 0\,,$$

$$\lambda_\pm = \tfrac{1}{2}v \pm \sqrt{n}\,, \qquad V_+ = 3\lambda_+ + \lambda_-\,, \quad V_- = 3\lambda_- + \lambda_+\,.$$

Equivalent initial conditions in terms of $\lambda_{\pm} = \frac{1}{2}v \pm \sqrt{n}$

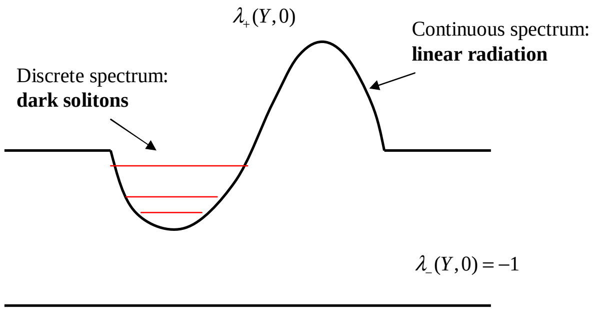


Projection, along the “dispersionless” characteristics, of the data at the piston onto Y-axis yields:

$$\lambda_{-}^{0} = -1\,,$$

$$\lambda_{+}^{0}(Y): \qquad \lambda_{+}^{0} = \tfrac{1}{2}f'(\tau) + 1, \quad Y = f(\tau) - (\tfrac{3}{2}f'(\tau) + 2)\tau.$$

Large T wave distribution: the semi-classical limit of the Zakharov-Shabat scattering problem



Discrete spectrum: generalized **Bohr-Sommerfeld quanization rule** for the defocusing NLS equation (*Jin, Levermore & McLaughlin (1999); Kamchatnov, Kraenkel, Umarov (2002)*):

$$\oint \sqrt{(\lambda_k-\lambda_+^0)(\lambda_k-\lambda_-^0)}\,dY = 2\pi(k+\tfrac{1}{2}), \quad k=0,1,\ldots,K,$$

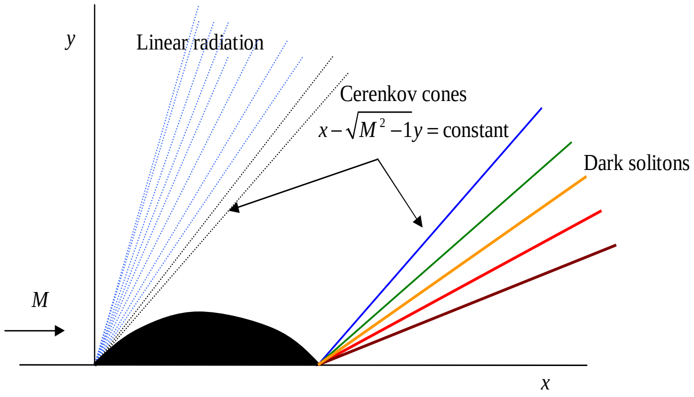
Density profile in the "dark" λ_k- spatial soliton:

$$n_k(x,y) = 1 - \frac{1-\lambda_k^2}{\cosh^2[\sqrt{1-\lambda_k^2}(y-(\lambda_k/M)x)]}.$$

Position of the soliton centres in the upper xy-plane are given by

$$y = (\lambda_k/M)x, \quad k=0,1,\ldots,K.$$

Reconciliation



- Two dispersive shocks are formed as intermediate wave states for finite x
- Asymptotic behaviour as $x \longrightarrow \infty$:
  - the leading dispersive shock wave transforms into a linear wave packet (“ship waves”).
  - the trailing dispersive shock wave transforms into oblique dark soliton fan.
- Full asymptotic description of the arising wave pattern as $M \longrightarrow \infty$, $\alpha \longrightarrow 0$, $M\alpha = \mathcal{O}(1)$ — is available in the framework of the single-phase Whitham modulation theory for the defocusing 1D NLS equation: general solution constructed in *A. Gurevich, A. Krylov and GE (1992), GE & A. Krylov (1995)*

Conclusions

- Supersonic flow of a BEC past an obstacle is very rich phenomenologically
- Two main ingredients of the asymptotic wave pattern for sufficiently large M:
  - Oblique dark solitons – inside the Mach cone. The stability has been established numerically
  - Small-amplitude “ship waves” – outside the Mach cone
- The two contrasting wave patterns above are reconciled in the asymptotic setting as the “long-distance” outcomes of two dispersive shock waves generated in the highly supersonic NLS flow past slender body.
- Optical counterparts of the described wave patterns can be observed and described analytically (work in progress)