Existence of (Generalized) Breathers in Periodic Media

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1. Models for 1D Photonic Crystals

$$\partial_x^2 u = \partial_t^2(\chi_1 u + \chi_3 |u|^2 u) \tag{1}$$

with $\chi_j(x) = \chi_j(x+L)$ for an $L > 0, x \in \mathbb{R}, t \geq 0$, and $u(x,t) \in \mathbb{R}$.

From a modeling point of view this quasi-linear equation is equivalent to the semi-linear model

$$\partial_t^2 u = \partial_x^2 u + \chi_1 u + \chi_3 |u|^2 u \tag{2}$$

2. Vanishing Group Velocities in Photonic Crystals

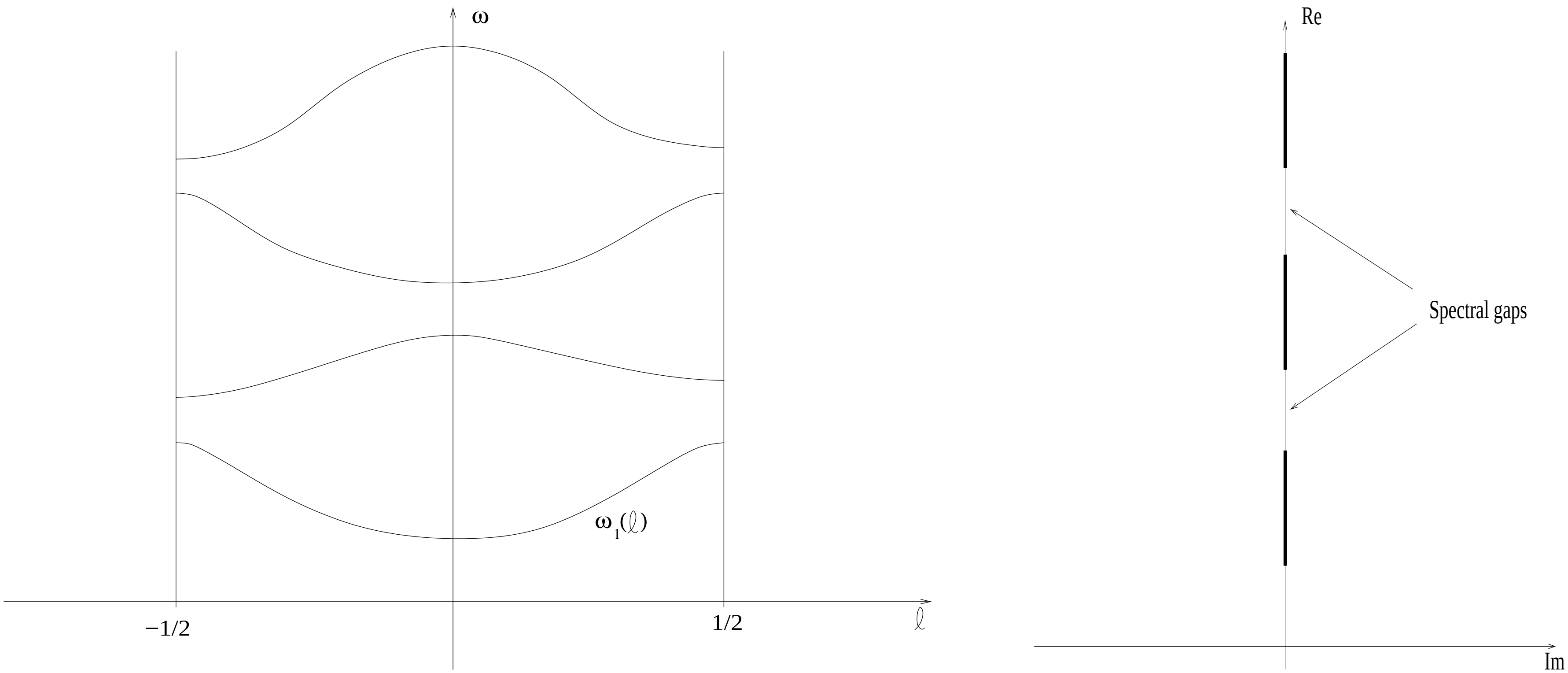

Figure 1: The eigenfunctions of $\partial_x^2 u + \chi_1 u = -\omega_n^2(\ell) u$ are given by $w_n(x,\ell)e^{i\ell x}$ with $w_n(x,\ell) = w_n(x+L,\ell)$. The right picture shows the curves $\ell \mapsto \omega_n^2(\ell)$; the left picture the spectrum with the spectral gaps as a subset of $\mathbb{C}$.

3. Standing Light Pulses in Photonic Crystals

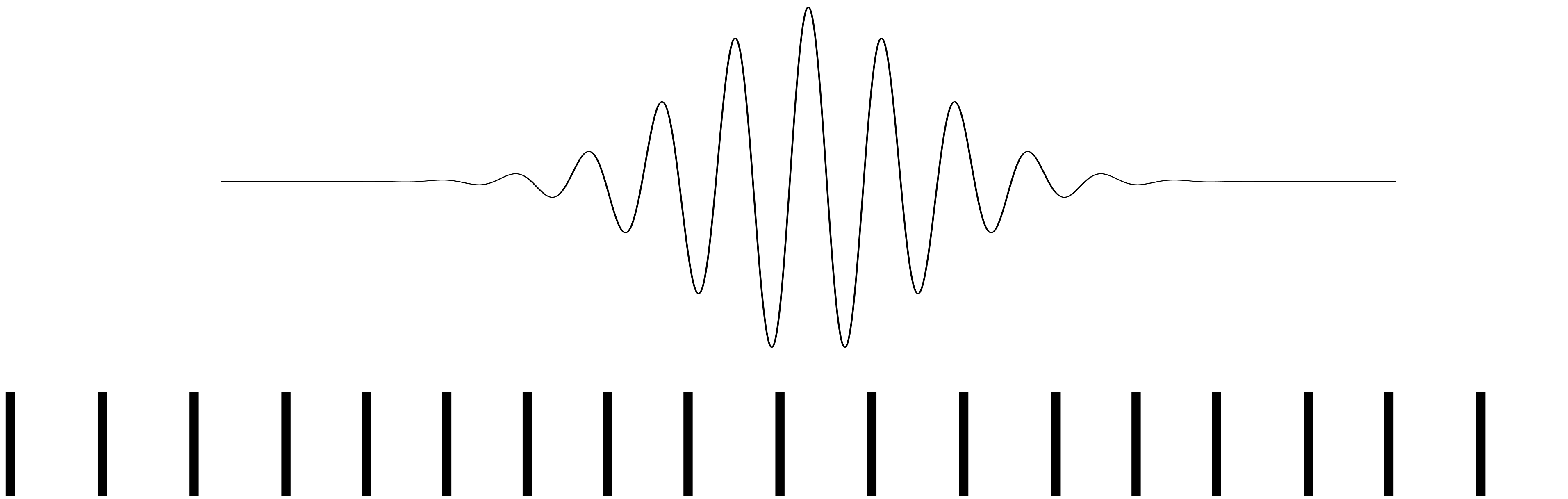

Figure 2: A standing light pulse in a photonic crystal. The wavelength of the carrier wave and the period of the photonic crystal are of a comparable order.

4. Rigorous Construction by Spatial Dynamics

spatial dynamics formulation

$$\partial_x^2 u = \partial_t^2 u - \chi_1 u - \chi_3 |u|^2 u, \tag{3}$$

look for homoclinic solutions in the phase space of $2\pi/\omega$-time periodic solutions

The linearized problem is solved by

$$u(x,t) = \mathrm{e}^{\mathrm{i}m\omega t} \mathrm{e}^{\lambda_m x} w_m(x)$$

with $w_m(x) = w_m(x + 2\pi)$ for $m \in \mathbb{Z}$ due to the periodic boundary conditions w.r.t. t and according to the theorem of Floquet.

5. The Spectral Properties

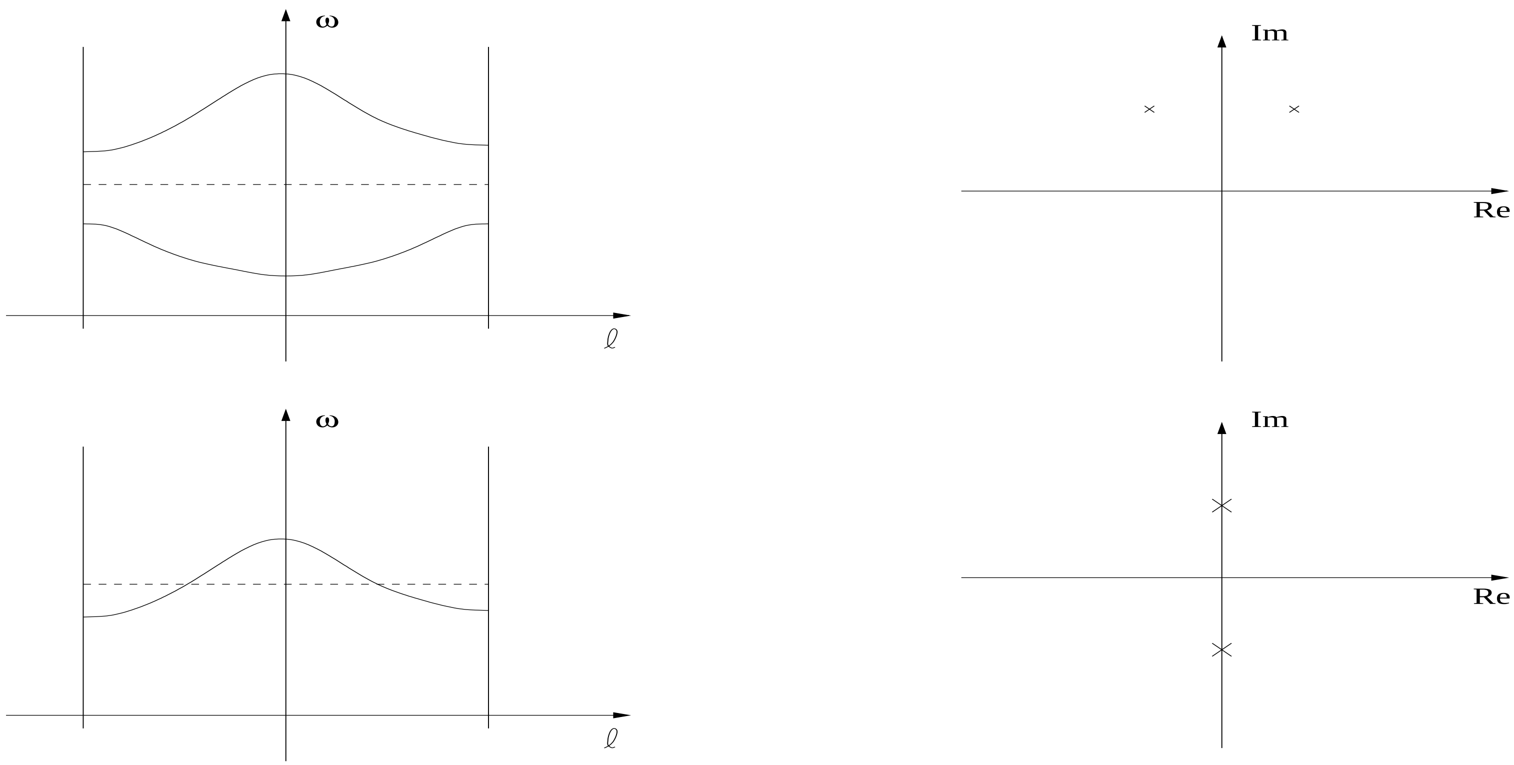

Figure 3: Solutions $u(x,t) = \mathrm{e}^{\mathrm{i}m\omega t}\mathrm{e}^{\lambda_m x} w_m(x)$. If the dotted line $\ell \mapsto n\omega$ falls into a spectral gap of the spectral picture of the time-dependent problem, then in the spectral picture of the spatial dynamics formulation two Floquet exponents off the imaginary axis occur. In the other case they are on the imaginary axis.

6. The Spectral Picture

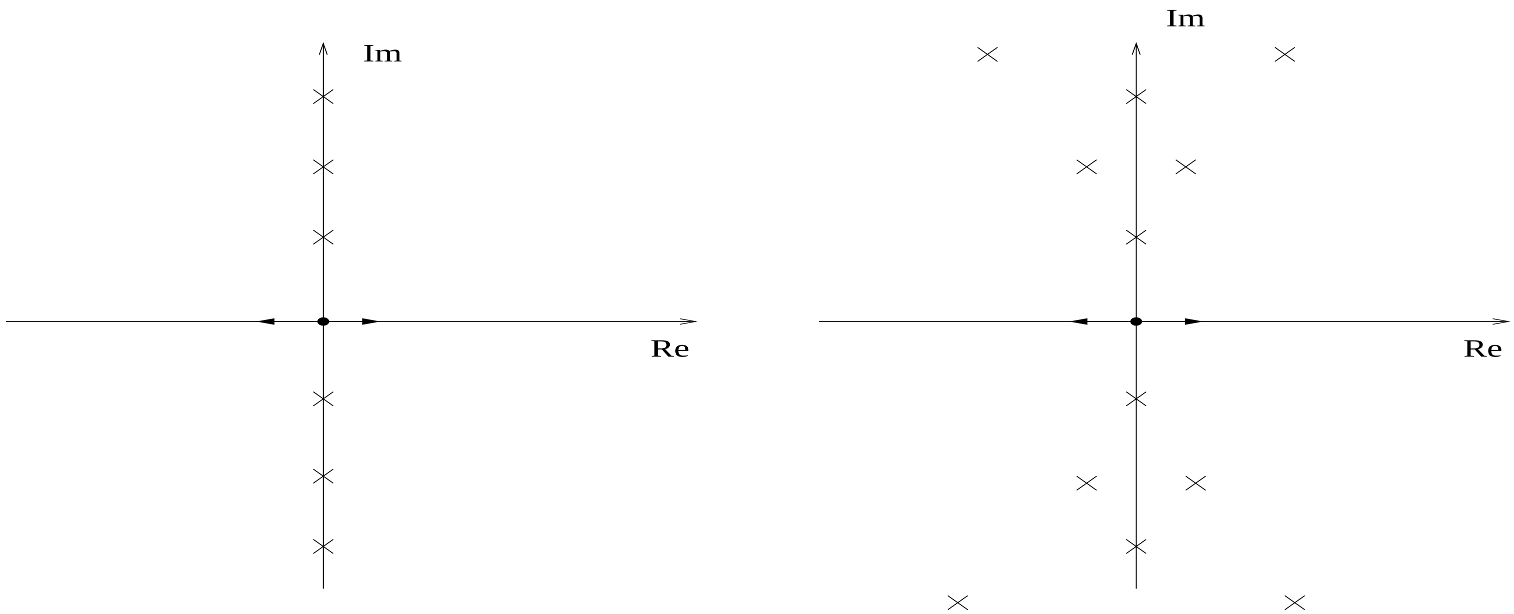

Figure 4: The eigenvalues and the Floquet exponents, respectively, of the spatial dynamics formulation in the homogeneous case in the left panel and the spatially periodic case in the right panel.

homogeneous case: [GS01CMP,GS05JDE,GS06Preprint]

7. The Result

Theorem: Let $\chi_j \in C_b^2$ with $\chi_j(x) = \chi_j(-x)$ for $j = 1,3$ and assume that $\lambda_j \notin \mathbb{Z}$ for $|j| \leq N$ for an $N \in \mathbb{N}$ fixed. Moreover, let $\omega = \omega_1(0) - \varepsilon^2$. Then there exist an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ Equation (2) possesses modulating standing pulses $u(x,t) = v(x,t)$ with

$$v(x,t) = v(-x,t), \quad v(x,t) = v\left(x, t + \frac{2\pi}{\omega}\right)$$

and

$$\sup_{x \in [-\varepsilon^{1-N}, \varepsilon^{1-N}]} |v(x,t) - h(x,t)| \leq C\varepsilon^{N-1},$$

where

$$\lim_{|x| \to \infty} h(x,t) = 0$$

and

$$\sup_{x \in \mathbb{R}} \left| h(x,t) - \left(\varepsilon\gamma_1 \mathrm{sech}(\varepsilon\gamma_2 x) w_1(x) \mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}\right) \right| \leq C\varepsilon^2 \tag{4}$$

with constants γ_1, γ_2 defined subsequently.

8. Sketch of a Generalized Breather

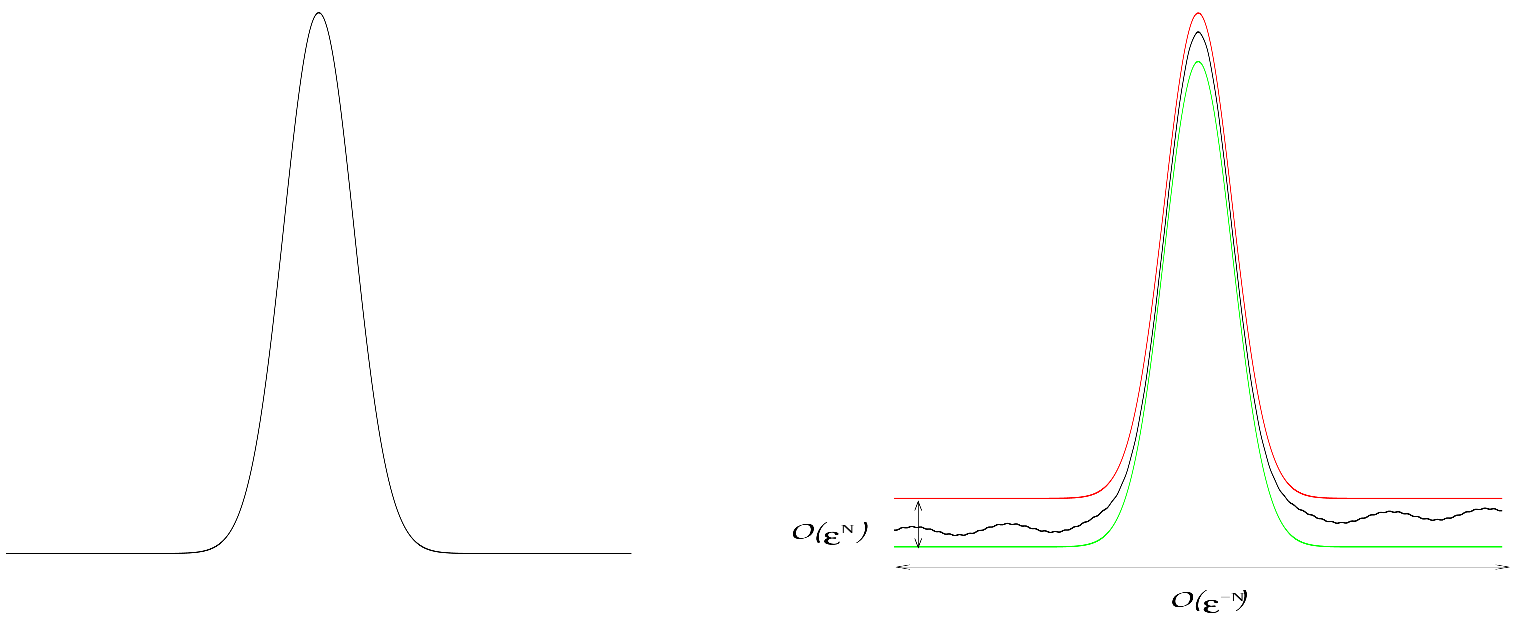

Figure 5: In the left panel we have modulating pulse solutions with $u \to 0$ for $|x| \to \infty$. In the right panel we have a generalized modulating pulse solution with $\mathcal{O}(\varepsilon^{N-1})$ tails for $|x|$ large as constructed in the theorem.

9. The Normal Form Transform

We split $u = w + q$ where q belongs to two zero Floquet exponents and w to the rest, i.e., we write (3) as

$$\begin{aligned} \partial_x q &= Mq + F(q,w), & (5) \\ \partial_x w &= Lw + G(q,w) + H(q), & (6) \end{aligned}$$

with M and L being linear operators and F, G, H being nonlinear operators with $G(q,0) = 0$.

If $H(q)$ vanished, then $\{w = 0\}$ would be some invariant subspace and

$$\partial_x q = Mq + F(q,0) \qquad (7)$$

would be some reduced four-dimensional system.

Obviously, $H(q) \neq 0$. A term $\mathcal{O}(|q|^n)$ in H can be eliminated in the equations for w if the non resonance condition

$$\lambda_j \notin \mathrm{i}\mathbb{Z} \qquad (8)$$

for $|j| \leq n$ is satisfied.

10. Discussion of Reduced System

Two-dimensional system

$$\partial_x q = Mq + F(q, 0; x) \tag{9}$$

Using the theorem of Floquet $\partial_x q = Mq$ is solved by

$$q(x) = \mathrm{e}^{Bx} P(x)(0)$$

with $P(x) = P(x + 2\pi)$. The transform $q(x) = P(x)\tilde{q}(x)$ leads to

$$\partial_x \tilde{q} = B\tilde{q} + P^{-1}(x) F(P(x)\tilde{q}, 0; x) \tag{10}$$

with B being a Jordan-block for $\varepsilon = 0$. The averaged system

$$\partial_x \check{q} = B\check{q} + \check{F}(\check{q}, 0)$$

possesses a homoclinic solution due to the reversibility. It is associated with the NLS-pulse.

Reversibility allows us to prove the persistence of the homoclinic solution for (10).

11. Persistence in the Full System

Unperturbed system

$$\partial_x q = Mq + F(q,0), \qquad \partial_x w = Mw \tag{11}$$

Full system:

- Construction of a center-stable manifold W_{cs} for (10)
- W_{cs} intersects fixed space of reversibility transversally
- Therefore, intersection persists in the full system
- Homoclinic form for $x \in [-\varepsilon^{1-N}, 0]$
- Use reversibility to contruct solutions also for $x \in [0, \varepsilon^{1-N}]$
- Technical detail:
  Split spectrum $= \{\operatorname{Re}\lambda < -\varepsilon^{N-1}\} \cup \{|\operatorname{Re}\lambda| \leq -\varepsilon^{N-1}\} \cup \{\operatorname{Re}\lambda > \varepsilon^{N-1}\}$.

12. Extensions

- Improvement: exponentially small tails [GS06Pre], not clear
- Complements: non-persistence of breathers [Denzler93]
- in photonic crystals breathers possible [Blank et al 07]