Finite-dimensional model for defect-trapped light in planar periodic nonlinear structures

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We study the dynamics of 2D gap solitons (GSs) in Bragg resonant nonlinear (photonic) gratings in the presence of localized defects. Previous work [Stud. Appl. Math. **115**, 209 (2005)] explains the mechanism of trapping the GS-carried energy at a defect via a resonant energy transfer from the GS into defect modes. We derive a finite-dimensional model that describes the evolution of the defect-trapped state as an interaction of linear defect modes and show that this model approximates the full dynamics very well in the regime when moderate amounts of GS energy are trapped. © 2006 Optical Society of America

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Pulse propagation in nonlinear periodic photonic structures is very promising for a set of applications in all optical processing.[1] A fundamental question is whether the combination of nonlinearity and periodicity allows for self-guided solutions. When the frequency of these lies in the photonic bandgap, they are called gap solitons (GSs). Most theoretical and experimental work has dealt with spatial solitons in the case where the incident beam is transverse to the direction of periodicity of the medium[2] (its refractive index) and with the analogous discrete solitons in waveguide arrays.[3] When the periodic structure is the Bragg grating oriented in the direction of propagation and a slowly varying envelope approximation applies, great progress has been achieved both in the theory[4] and experiments[5] to describe light propagation in 1D nonlinear geometries (e.g., fiber gratings or periodic semiconductor waveguides). In fact, closed-form GSs for the governing coupled-mode system have been found.[4] However, much less is known when the waveguide is not fully confined in the transverse direction; Ref. 6 and the X-symmetry case of Ref. 7 present a study of the possible existence and robustness of the GSs in one time and two spatial dimensions in a Kerr nonlinear waveguide, which prevents diffraction in one transverse direction (y) via total internal reflection, is homogeneous in the other transverse direction (x), and possesses a weak Bragg resonant grating in the propagation direction (z). Analogous to the 1D GS,[4] the frequency of these GSs lies inside the (incomplete) frequency bandgap. They are long lived and, unlike localized pulses in homogeneous 2D Kerr media, have a tunable velocity v with $|v|<c_g$, where c_g is the group velocity and under perturbations does not undergo blowup in finite time.[8]

Recent research considers possible ways to trap, reroute, switch solitons or solitonlike pulses by, for example, adding defects in the photonic structure, see, e.g., Ref. 9 for experiments in AlGaAs waveguides. Important questions are the nature of the interaction of incident solitons with defects and the dynamics of energy trapped in the intrinsic defect modes. Here, we address the latter question for interactions of GS, with defects in the above-described 2D waveguide of Ref. 6, see Fig. 1. Pulse dynamics in this geometry is well described by the coupled-mode equations:

$$[i\partial_t \pm i\partial_z + \partial_x^2 + V(x,z)]E_\pm + \kappa(x,z)E_\mp + N_\pm = 0, \quad (1)$$

where $E_\pm$ are the forward and backward propagating wave-packet envelopes, respectively, $N_\pm=\Gamma(|E_\pm|^2+2|E_\mp|^2)E_\pm$, and κ, $\Gamma>0$.

In Eq. (1), the group velocity c_g has been normalized to 1. In the absence of defects, $V\equiv 0$ and $\kappa\equiv$constant, GSs of the form $[E_+,E_-]=e^{-i\omega t}\times[\mathcal{E}_+(x,\zeta),\mathcal{E}_-(x,\zeta)]$, $\omega\in\mathbb{R}$, where $\zeta=z-vt$ and $\omega=\omega(v)=\omega_0\sqrt{1-v^2}$, have been found numerically[6] for frequencies near the upper edge of the incomplete bandgap, i.e., when $\omega_0 \lessapprox \kappa$, and for all velocities $v\in(-1,1)$.

The defect (a local variation of the linear index of refraction) is modeled by the (smooth) functions V and κ. If such a defect creates a lensing effect, localized stationary solutions, called defect modes, of Eq. (1) of the form $e^{-i\eta t}[\psi_+(x,z),\psi_-(x,z)]$, $\eta\in\mathbb{R}$ exist. As in similar systems, linear defect modes ($\Gamma=0$) give rise to bifurcation into nonlinear modes ($\Gamma\neq 0$).[6,10] Previously,[10] a mechanism predicting the nature of interactions (transmission, reflection, or trapping) of 1D GSs with localized defects has been devised based on the principle of a resonant energy transfer from the GS to the defect mode(s). According to this mechanism, a slow enough GS with a frequency of

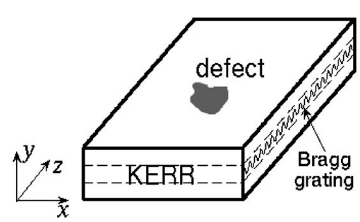

Fig. 1. Waveguide.

$\omega = \omega_1$ is trapped when a nonlinear defect mode with the same frequency of $\eta = \omega_1$ and smaller total power[11] $\|\psi_+\|^2_{L^2(\mathbb{R}^2)} + \|\psi_-\|^2_{L^2(\mathbb{R}^2)}$ than that of the GS exists. In Ref. 6, this principle is demonstrated to also hold for the analogous 2D GS; a number of simulated interactions show behavior ranging from partial transfer to no transfer of energy into the defect modes.

For the case where the defect modes are excited, the dynamics that follows was not studied in any detail. In this Letter, we show how in cases when the amount of trapped energy is relatively small the energy is stored in linear defect modes and the dynamics of the trapped state can be well approximated by a finite-dimensional model that describes the evolution of the activated modes, which validates the resonant energy transfer principle. The finite-dimensional system models the dynamics after trapping at an incomparably smaller numerical expense than the full formulation, Eq. (1). Below we derive the governing equations explicitly for the case of two resonant defect modes and present numerical simulations to demonstrate the behavior.

Suppose a given defect V, κ supports M linear defect modes $e^{(i\eta_{L_k}t)}(\psi_{+_k}, \psi_{-_k})^T$, $k=1,\ldots,M$ with frequencies η_{L_k}. If the total trapped power is small, the solution after trapping is approximated by a linear combination of the corresponding linear defect modes, i.e.,

$$\begin{bmatrix} E_+(x,z,t) \\ E_-(x,z,t) \end{bmatrix} \approx \sum_{k=1}^{M} a_k(t) e^{-i\eta_{L_k}t} \begin{bmatrix} \psi_{+_k}(x,z) \\ \psi_{-_k}(x,z) \end{bmatrix}. \tag{2}$$

To find the evolution of the coefficient $a_k(t) \in \mathbb{C}$, we substitute approximation (2) in Eq. (1) and perform the inner product with $(\psi_{+_k}, \psi_{-_k})^T$. Due to the orthonormality of the modes, one obtains a set of M ordinary differential equations coupled only via the nonlinear terms. Their form is $ia_k' + (\mathrm{NL})_k = 0$, where $(\mathrm{NL})_k = \int_\Omega \psi^*_{+_k}(\mathrm{NL})_+ + \psi^*_{-_k}(\mathrm{NL})_- \mathrm{d}x\mathrm{d}z$ and $(\mathrm{NL})_\pm$ come from the substitution of approximation (2) into the nonlinear part $N_\pm$ of Eq. (1).

For the case of two defect modes ($M=2$) under the symmetry assumptions $\psi_{+_k}(x,-z) = \psi^*_{+_k}(x,z)$ and $\psi_{-_1} = -\psi^*_{+_1}$, $\psi_{-_2} = \psi^*_{+_2}$ (or $\psi_{-_1} = \psi^*_{+_1}$, $\psi_{-_2} = -\psi^*_{+_2}$), the equations are

$$i\tilde{a}_1' - \Delta\eta\tilde{a}_1 + \mu_1|\tilde{a}_1|^2\tilde{a}_1 + 2\nu|\tilde{a}_2|^2\tilde{a}_1 + \nu\tilde{a}_2^2\tilde{a}_1^* = 0,$$

$$i\tilde{a}_2' + \Delta\eta\tilde{a}_2 + \mu_2|\tilde{a}_2|^2\tilde{a}_2 + 2\nu|\tilde{a}_1|^2\tilde{a}_2 + \nu\tilde{a}_1^2\tilde{a}_2^* = 0, \tag{3}$$

where $\tilde{a}_1 = a_1 e^{-i\Delta\eta t}$, $\tilde{a}_2 = a_2 e^{-i\Delta\eta t}$, $\Delta\eta = (\eta_{L_1} - \eta_{L_2})/2$, $\mu_k = \int_\Omega |\psi_{+_k}|^4 \mathrm{d}x\mathrm{d}z$, and $\nu = \int_\Omega 4|\psi_{+_1}|^2|\psi_{+_2}|^2 + 2\Re(\psi^2_{+_1}\psi^{*2}_{+_2}) \times \mathrm{d}x\mathrm{d}z$. The solutions (a_1, a_2) can be expressed in terms of Jacobian elliptic functions[12] and evolve on the circle $|a_1|^2 + |a_2|^2 = \text{constant}$.

We next perform numerical simulations of the full model (1) for GS–defect interactions. The defects are chosen so that they have two linear defect modes close to resonance with the corresponding stationary GSs with parameter values to allow energy transfer from the GSs into the modes. To verify the validity of Eqs. (3), its solutions a_1, a_2 are compared with the projections[11] $A_k(t) = \langle E_+, \psi_{+_k}\rangle + \langle E_-, \psi_{-_k}\rangle$, $k=1,2$, respectively, of the full solution (E_+, E_-). The comparison is done for $t \geqslant t_1$, where t_1 is a time after the interaction when radiation has left the defect region and the trapped energy has a stationary (or periodic) profile. To fully describe the behavior, we also study the evolution of the power[11] $\mathcal{N}(t) = \|E_+\|^2_{L^2(\Omega)} + \|E_-\|^2_{L^2(\Omega)}$ and of the energy $\mathcal{N}_{\mathrm{dm}}(t) = |A_1|^2 + |A_2|^2$ contained in the linear defect modes. For a better visualization, we also plot the time evolution of $|E_+|$ along the x coordinate corresponding to the defect center.

For the simulations of Eq. (1), we use a finite-difference time-domain method with time stepping via a fourth-order additive Runge–Kutta (ESDIRK) scheme,[13] treating only the second derivative term implicitly, and a fourth-order central difference approximation of the derivatives in space. As the study requires simulations of long time evolution, the treatment of outgoing radiation is crucial. By implementing the perfectly matched boundary layer method,[6,14,15] we prevent radiation from being artificially reflected at the domain boundary and from affecting the dynamics at the defect location. Examples that highlight the validity and limitations of Eqs. (3) follow.

Example 1. GSs with the velocity $v=0.3$ and frequencies of the corresponding stationary GS being $\omega_0 \approx 0.931$ and $\omega_0 \approx 0.959$ are partly trapped at the defect modeled by $V = V1 + V2/V3$, with $V_1 = 2\beta^2 \operatorname{sech}^2(\beta x) T(z;9)$, $V_2 = k^2\sqrt{\kappa_\infty^2 - k^2} \operatorname{sech}^2(kz) T(x;7)$, $V_3 = 2(\kappa_\infty^2 + k^2[\tanh^2(kz) - 1])$ and with $\kappa = T(x;7)\sqrt{\kappa_\infty^2 + k^2[\tanh^2(kz)-1]}$, where $T(y;d) = \frac{1}{2}[\tanh(y+d) - \tanh(y-d)]$, $(k,\beta,\kappa_\infty) = (0.18, 0.16, 1)$. In the definition of A_k and $\mathcal{N}$, we use $\Omega = [-36, 36] \times [-31, 31]$.

Figure 2(c) shows good agreement between the finite-dimensional model initialized at $t_1 = 400$ and the full dynamics in the case of $\omega_0 \approx 0.959$ where a smaller amount of energy is trapped. This amount is, nevertheless, $\sim 15\%$ of the original power $\mathcal{N}(0)$ in the GS, which is a nonnegligible portion and may prove usable in signal processing applications. The agreement is slightly worse for $\omega_0 \approx 0.931$, where more energy is trapped. Also note that the trapped state for the GS with $\omega_0 \approx 0.959$ is dominated by the mode number 2 [$|A_2(t)| > |A_1(t)|$ for $t > 150$] although ω_0 is closer to η_{L_1}. This is likely caused by the adiabatic nature of the trapping process (see the end of Example 2) during which the GS perturbs into one with a higher frequency. The GS with $\omega_0 \approx 0.931$ is further from resonance with the two modes, and their contributions are approximately balanced. The apparent wobbling of the trapped state in plots 2(a) is due to the coexistence of two defect modes rather than a back and forth movement of a trapped pulse.

Example 2. A GS with $\omega_0 \approx 0.9595$ is sent toward a Gaussian defect $V = 0.3e^{-(ax^2+bz^2)}$, $\kappa = 1 + 0.1e^{-(ax^2+bz^2)}$ with $b = 2a = 0.04$. This defect supports two linear modes with $\eta_{L_1} \approx 0.9048$ and $\eta_{L_2} \approx 0.9805$. We use

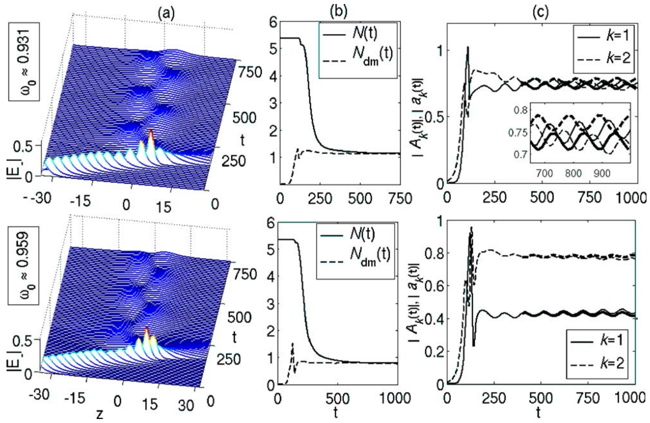

Fig. 2. (Color online) Example 1: Top and bottom rows $\omega_0 \approx 0.931$ and 0.959, respectively. Column (a) $|E_{-}(0,z,t)|$ ($|E_{+}|$ qualitatively the same); (b) (—) $\mathcal{N}(t)$, (--) $\mathcal{N}_{\mathrm{dm}}(t)$; (c) Eq. (1) vs. Eq. (3). Regular lines, $|A_k(t)|$; bold lines, $|a_k(t)|$.

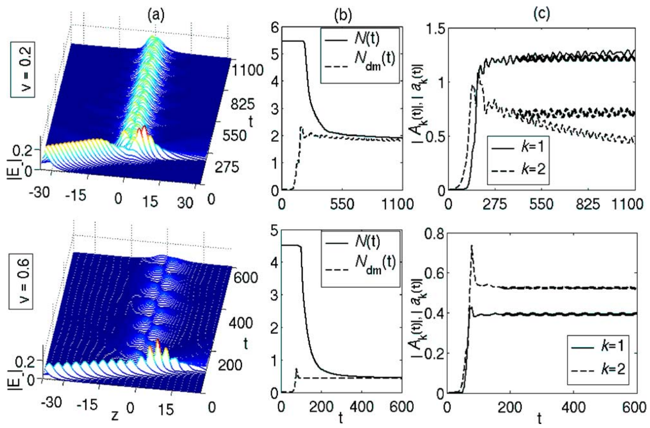

Fig. 3. (Color online) Example 2. Top and bottom rows, $v=0.2$ and 0.6, respectively. Column (a) $|E_{-}(0,z,t)|$ ($|E_{+}|$); (b) (—) $\mathcal{N}(t)$, (--) $\mathcal{N}_{\mathrm{dm}}(t)$; (c) Eq. (1) vs. Eq. (3), regular lines: $|A_k(t)|$, bold lines: $|a_k(t)|$.

here $\Omega=[-38,38]\times[-45,45]$. Figure 3 shows two incident velocity cases, $v=0.2$ and 0.6, and clearly, for the faster GS, less energy is trapped and the dynamics after trapping is approximated by Eqs. (3) better than in the slower case, where the trapped state is more nonlinear. In the two cases of this example, $t_1=400$ and 170, respectively. Column (c) of Fig. 3 also shows that while for $v=0.2$ the mode with the smaller frequency η_{L_1} is dominant, for $v=0.6$ it is the higher-frequency mode. This, we conjecture, is due to the adiabatic nature of the interactions. As the GS encounters a defect, it starts losing energy to radiation, and, if in the initial stages of the interaction dynamics it remains a GS, its frequency has to increase, since the GSs satisfy $\partial\mathcal{N}/\partial\omega<0$. As faster GSs lose more energy, this explains their stronger coupling into defect modes with higher frequencies.

In conclusion, we have investigated the interaction of (spatiotemporal) GSs in 2D Kerr nonlinear Bragg gratings with localized defects and verified that for the case of a relatively low energy trapping (up to 15% energy stored in the cases studied) the dynamics after trapping is very well approximated by a finite-dimensional model describing the interaction of the linear defect modes. Its dimension is therefore equal to the number of defect modes activated via resonance in the trapping process. When the amount of trapped energy is larger, the trapped state seems to be a truly nonlinear defect mode and the derived model does not apply. Confirming that, in the former case, the stored energy takes the form of linear or weakly nonlinear modes is an important theoretical result. In light of continued research on photonic structures,[16] our classical model and numerical scheme can be advanced to describe experiments in practical devices, e.g., optical memory, where slow light in the form of a GS is trapped in an optical mode (state).

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